Intermittency and the convergence of integrated supOU processes

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Outline

- **Intermittency:**
  - We shall define it.
  - We will show that a self-similar process cannot be intermittent.
  - But how does this fit with Lamperti's theorem which states that if we have convergence of normalized sums, then the limit is self-similar?
- **We shall use supOU processes as our test cases.**
  - I will introduce these processes
  - It will be like bringing coal to Newcastle
- **We will show that integrated supOU processes can be intermittent.**
- **We will then focus on limit theorems**
- **Conclusion: A process can be intermittent and satisfy a limit theorem**
- **Conclusion: Intermittency involves an unusual behavior of the moments.**

Lamperti's theorem

Let $X(t), t \geq 0$ be a strictly stationary process and suppose without loss of generality that it has mean zero. Let $X^*(t) = \int_0^t X(s)ds, t \geq 0$ be the aggregated process. Suppose that

$$\left\{ \frac{X^*(Tt)}{A_T} \right\} \xrightarrow{d} \{Z(t)\}, \tag{1}$$

as $T \to \infty$ with convergence in the sense of convergence of all finite dimensional distributions as $T \to \infty$. By Lamperti's theorem, the normalizing sequence is always of the form $A_T = L(T)T^H$ for some $H > 0$ and L slowly varying at infinity. Moreover, the limiting process X^* is H-self-similar, that is, for any $c > 0$,

$$\{Z(ct)\} \stackrel{d}{=} \{c^H Z(t)\},$$

where $\{\cdot\} \stackrel{d}{=} \{\cdot\}$ denotes the equality of finite dimensional distributions.

The scaling function and intermittency

For a process $X^* = \{X^*(t),\ t \geq 0\}$, let $(0, \overline{q}(X^*))$ denote the range of finite moments, that is

$$\overline{q}(X^*) = \sup\{q > 0 : \mathbb{E}|X^*(t)|^q < \infty \ \forall t\}.$$

Definition. The **scaling function** at point $q \in (0, \overline{q}(X^*))$ of the process X^* is

$$\tau_{X^*}(q) = \lim_{t \to \infty} \frac{\log \mathbb{E}|X^*(t)|^q}{\log t}.$$

Definition. A stochastic process $X^* = \{X^*(t),\ t \geq 0\}$ is **intermittent** if there exist $q_1 < q_2 \in (0, \overline{q}(X^*))$ such that

$$\frac{\tau_{X^*}(q_1)}{q_1} < \frac{\tau_{X^*}(q_2)}{q_2}. \tag{2}$$

Note. If X^* itself is H-self-similar, then

$$\tau_{X^*}(q) = Hq, \ \ q \in (0, \overline{q}(X^*))$$

that is, $\tau_{X^*}(q)$ is linear in q. Hence a self-similar process cannot be intermittent.

Important note

Consider the relation

$$\left\{ \frac{X^*(nt)}{A_n} \right\} \xrightarrow{d} \{Z(t)\}.$$

We want to study the intermittency of X^* and not of the limit $Z(t)$ because that limit is self-similar and hence cannot be intermittent.

What does Lamperti's theorem imply on the scaling function $\tau_{X^*}(q)$?

Theorem

Let $X^* = \{X^*(t),\, t \geq 0\}$ and $Z = \{Z(t),\, t \geq 0\}$ be two processes such that $Z(t)$ is nondegenerate for every $t > 0$ and suppose that for a sequence (A_n), $A_n > 0$, $\lim_{n\to\infty} A_n = \infty$, one has

$$\left\{ \frac{X^*(nt)}{A_n} \right\} \xrightarrow{d} \{Z(t)\}, \tag{3}$$

with convergence in the sense of convergence of all finite dimensional distributions as $n \to \infty$. Then there exists a constant $H > 0$ such that for every $q > 0$ satisfying

$$\frac{\mathbb{E}|X^*(nt)|^q}{A_n^q} \to \mathbb{E}|Z(t)|^q, \quad \forall t \geq 0, \tag{4}$$

the scaling function of X^* at q is

$$\tau_{X^*}(q) = Hq. \tag{5}$$

Therefore, in the intermittent case either (3) or the convergence of moments (4) fail or both must fail to hold. We will show:

Theorem

Integrated supOU processes can be intermittent. (Also true for trawl processes.)

The supOU process

The supOU process will be defined through successive steps:

1. $$dX(t) = -\lambda X(t)dt + dB(\lambda t), \qquad \lambda > 0, t \geq 0.$$

   SDE, $B(t)$ is Brownian motion, Mean reversion to the origin through $-\lambda$.

2. Integral form. $X(t)$ is strictly stationary.

   $$X(t) = e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} dB(\lambda s) = \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dB(s), \qquad \lambda > 0, t \geq 0.$$

3. $$X(t) = \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s), \qquad \lambda > 0, t \geq 0.$$

   $L(s)$ is a Lévy process with $\mathbb{E} \log(1 + |L(1)|) < \infty$, so that $X(t)$ is well defined. $L(s)$ is independently scattered, has stationary increments and $L(1)$ is infinitely divisible.

4. Randomize λ using the probability distribution π.

   $$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s) d\pi(\lambda) \qquad t \geq 0.$$

How does the mixing measure π affects things?

π does not affect the marginal distribution of $X(t)$ because $X(t)$ is stationary and the representation of the process involves λt:

$$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s) d\pi(\lambda) \qquad t \geq 0.$$

But π affects the dependence structure.

Modeling considerations

The supOU process is attractive because one can model

- the dependence though the non-random mixing measure π.
- the marginal distribution through the random Lévy process $L(s)$:

$$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s) d\pi(\lambda) \qquad t \geq 0.$$

This is attractive, particularly in finance.

Recall:

Lévy process

$= \text{drift} + \text{Gaussian component} + \text{pure jump Lévy component}$

The pure jump Lévy component is a limit of compound processes.

How does the mixing measure π affects the dependence structure?

Assume that $X(t)$ has finite variance. Then its scaling function is given by

$$r(\tau) = \int_{\mathbb{R}_+} e^{-\tau\lambda}\pi(d\lambda), \quad \tau \geq 0, \tag{6}$$

that is, the correlation is the Laplace transform of π. Hence we have:

Proposition

Suppose X is a square integrable supOU process with correlation function r, L is a slowly varying function at infinity and $\alpha > 0$. Then

$$\pi\left((0, x]\right) \sim L(x^{-1})x^{\alpha}, \qquad \text{as } x \to 0$$

if and only if

$$r(\tau) \sim \Gamma(1+\alpha)L(\tau)\tau^{-\alpha}, \qquad \text{as } \tau \to \infty.$$

Long-range dependence

We saw that for $\alpha > 0$,

$$\pi(0, x] \approx x^{\alpha}, \qquad \text{as } x \to 0$$

is equivalent to

$$r(\tau) \approx \tau^{-\alpha}, \qquad \text{as } \tau \to \infty.$$

If $\alpha \in (0, 1)$, then we have long-range dependence

because $\int_0^\infty r(\tau) = \infty$ and

$$\mathrm{Var}\ X^*(t) = \int_0^t \int_0^t r(u - v)dudv \approx t^{-\alpha+2} \text{as } t \to \infty.$$

If we set $\mathrm{Var}\ X^*(t) \approx t^{2H}$, we have $2H = -\alpha + 2$, with

$$1/2 < H < 1.$$

If $\alpha > 1$ we have short-range dependence

because

$$\mathrm{Var}\ X^*(t) = \int_0^t \int_0^t r(u - v)dudv \approx t \text{ as } t \to \infty,$$

and thus $2H = 1$ or $H = 1/2$.

Examples

1. $\pi(\{\lambda\}) = 1$
2. π is a discrete probability on $\lambda_k, k = 1, 2, \cdots$. Then let

$$X(t) \stackrel{d}{=} \sum_{k=1}^{\infty} X^{(k)}(t), \; t \geq 0,$$

where $\{X^{(k)}(t), \, t \in \mathbb{R}\}$, $k \in \mathbb{N}$ are independent OU type processes corresponding to parameter λ_k and its characteristic or cumulant function is weighted by p_k. From (6) the correlation function is

$$r(\tau) = \sum_{k=1}^{\infty} e^{-\lambda_k \tau} p_k, \quad \tau \geq 0.$$

3. π is a Gamma distribution:

$$\pi(dx) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \mathbf{1}_{(0,\infty)}(x) dx,$$

where $\alpha > 0$. Then $\pi((0, x]) \approx x^{\alpha}, \quad x > 0$. In fact

$$r(\tau) = (1 + \tau)^{-\alpha}$$

Long-range dependence if $\alpha \in (0, 1)$.

Analyticity assumption

We will assume that the cumulant function of $X(t)$

$$\kappa_Y(\theta) = C\{\theta \ddagger X\} = \log \mathbb{E} e^{i\theta X}$$

is analytic in a neighborhood of the origin in the complex plane.

- This ensures the existence of all the moments and cumulants of the marginal distribution of the underlying supOU process $X(t)$.
- In proofs we can use expansions of the cumulant function.
- We may then involve high moments when discussing intermittency (it may be enough to take derivatives up to finite order).
- The analyticity does not depend on the mixing measure π since the choice of π does not affect the marginal distribution of X.
- The following is a useful criterion for checking analyticity of the cumulant function:

Lemma

The characteristic and cumulant functions are analytic in a neighborhood of the origin if and only if there is a constant C such that the corresponding distribution function F satisfies

$$1 - F(x) + F(-x) = O(e^{-ux}), \quad \text{as } x \to \infty,$$

for some $u > 0$.

- It follows that the cumulant function of $X(t)$ is analytic in the neighborhood of the origin if there exists $a > 0$ such that $\mathbb{E} e^{a|X(t)|} < \infty$.

Example: the inverse Gaussian distribution

$IG(\delta,\gamma)$, $\gamma > 0$, $\delta > 0$. It has density

$$f_{IG(\delta,\gamma)}(x) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} \exp\left\{ -\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right\} \mathbf{1}_{(0,\infty)}(x)$$

Hence, there is $a > 0$ such that $\mathbb{E}e^{a|X(t)|} < \infty$, the cumulant generating function is analytic in a neighborhood of the origin and has the form

$$\kappa_X(\theta) = \delta\left(\gamma - \sqrt{\gamma^2 - 2i\theta}\right).$$

Example: the normal inverse Gaussian distribution

$$NIG(\alpha, \beta, \delta, \mu) \text{ with parameters } \alpha \geq |\beta|,\ \delta > 0,\ \mu \in \mathbb{R}$$

The density of $NIG(\alpha, \beta, \delta, \mu)$ distribution satisfies

$$f_{NIG(\alpha,\beta,\delta,\mu)}(x) \sim C|x|^{-3/2} e^{-\alpha|x|+\beta x}, \qquad \text{as } x \to \pm\infty.$$

Hence, there is $a > 0$ such that $\mathbb{E}e^{a|X(t)|} < \infty$, the cumulant generating function is analytic in a neighborhood of the origin and has the form

$$\kappa_X(\theta) = i\mu\theta + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\theta)^2}\right).$$

Other examples of supOU processes satisfying the required conditions can be obtained by taking the marginal distribution to be gamma, variance gamma, tempered stable, Euler's gamma.

Note: we can deal with the heavy-tailed student distribution by having the moments finite up to certain order.

Brief notation review

- X(t) is a supOU process. It is strictly stationary and all its moments are finite.
- $$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\lambda t + s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s) d\pi(\lambda) \qquad t \geq 0.$$
- $\pi(0, x] \approx x^{\alpha}, \quad$ as $x \to 0, \alpha > 0$.
- $\alpha \in (0,1)$ LRD, $\qquad \alpha > 1$ SRD.
- $X^*(t) = \int_0^t X(s) ds, t \geq 0$
- The **scaling function** of the process X^* at point q is
$$\tau_{X^*}(q) = \lim_{t \to \infty} \frac{\log \mathbb{E}|X^*(t)|^q}{\log t},$$
- A stochastic process $X^* = \{X^*(t),\ t \geq 0\}$ is **intermittent** if there exist $q_1 < q_2$ such that
$$\frac{\tau_{X^*}(q_1)}{q_1} < \frac{\tau_{X^*}(q_2)}{q_2}.$$
- If Z is a H-self-similar process with $\mathbb{E}|Z(t)|^q < \infty$, then $\tau_Z(q) = Hq$, and $\tau_Z(q)/q$ is constant, therefore the process is not intermittent.

The three processes

Do not confuse the following three processes:

- $X(t), t \geq 0$: supOU, stritly stationary, all moments finite, mean subtracted
- $X^*(t) = \int_0^t X(s)ds, t \geq 0$: stationary increments, all moments finite
- $Z(t), t \geq 0$: limit process, may have infinite variance

Intermittency is associated with $X^*(t)$.

The intermittency theorem.

Theorem

Let $X(t),\ t > 0$ be a non-Gaussian supOU process such that

- $\pi(0, x) \equiv L(x^{-1})x^{\alpha}$ *as* $x \to 0,\ \alpha > 0$.
- *The cumulant function of $X(t)$ is analytic around the origin.*
- $\mathbb{E}(X(t)) = 0, \mathbb{E}(X(t)^2) \neq 0$.

Then for every $q \geq q^*$,

$$\tau_{X^*}(q) = q - \alpha,$$

where q^* is the smallest even integer greater than 2α. Hence, for $q^* \leq q_1 < q_2$,

$$\frac{\tau_Y(q_1)}{q_1} = 1 - \alpha/q_1 < 1 - \alpha/q_2 = \frac{\tau_X^*(q_2)}{q_2},$$

so $X^* = \{X^*(t),\ t \geq 0\}$ is intermittent.

Note: "Non-Gaussian" means that the Lévy process includes a pure jump component.
Note on proof: cumulants $\to$ even moments $\to$ absolute moments $\to$ moments

Questions

Suppose that $X^* = \{X^*(t),\ t \geq 0\}$ is intermittent.

Does $X^*(Tt)$ adequately normalized converge in the sense of finite dimensional distribution?

- If yes, to what?
- How is this compatible with the intermittency?
- How does it fit with Lamperti's theorem?

The Lévy process defining $X(t)$ has a Gaussian component

Theorem

Suppose that the Lévy process defining $X(t)$ has a Gaussian component but is not purely Gaussian (i.e. it has also a jump component), and let $\mathbb{E}X(t) = 0$, $\sigma^2 = \mathrm{Var}X(t) < \infty$ and $\alpha \in (0,1)$ with some slowly varying function L.
Then as $T \to \infty$

$$\left\{ \frac{1}{T^{1-\alpha/2} L(T)^{1/2}} X^*(Tt) \right\} \xrightarrow{d} \{\widetilde{\sigma} B_H(t)\},$$

where $\{B_H(t)\}$ is fractional Brownian motion with $H = 1 - \alpha/2 \in (1/2, 1)$ and

$$\widetilde{\sigma}^2 = \sigma^2 \frac{\alpha}{2-\alpha} \int_0^\infty \left(1 - e^{-1/z}\right) z^{-\alpha} dz.$$

Remark: By the intermittency theorem, if the cumulant function of $X(t)$ is analytic around the origin, then $X^*(t)$ is intermittent.
Remark: The theorem holds also if $X(t)$ is purely Gaussian, but in that case

$$\mathbb{E}|N(0,\sigma^2)|^q = C\sigma^q$$

Then the process $X^*(t)$ cannot be intermittent.

Let's look at other limit theorems

Basic assumptions

To get intermittency for $X^* = \{X^*(t),\ t \geq 0\}$ in the sequel, we will always suppose:

- The cumulant function for $X(t)$ is analytic around the origin. In particular, all the moments are finite.
- $\mathbb{E}X(t) = 0, \mathbb{E}(X(t)^2) \neq 0$

The Lévy process defining $X(t)$ is a pure jump process

Thus we now suppose that the Lévy process does not have a Gaussian component.

Introduction of the parameter $\beta > 0$:

In addition to the dependence parameter α, the limit will depend on the behavior of the Lévy measure μ_L near the origin. We assume there exists $\beta > 0$, $c^+, c^- \geq 0$, $c^+ + c^- > 0$ such that

$$\lim_{x \downarrow 0} x^{\beta} \mu_L\left([x, \infty)\right) = c^+ \text{ and } \lim_{x \downarrow 0} x^{\beta} \mu_L\left((-\infty, -x]\right) = c^-. \tag{7}$$

Note that we must always have $\beta < 2$ since the Lévy measure μ_L must satisfy $\int_0 x^2 \mu_L(dx) < \infty$.

Lévy-stable limit

Theorem

Suppose that

- $\mathbb{E}X(t) = 0, \sigma^2 = \mathrm{Var}X(t) < \infty$
- π *involves* $\alpha \in (0,1)$ *and some slowly varying function* L.
- *There is no Gaussian component*
- *There is a* $\beta > 0$ *such that* $\int_{\mathbb{R}} x^{\beta} \mu_L(dx) < \infty$.
- $$0 < \beta < 1 + \alpha < 2.$$

Then, as $T \to \infty$,

$$\left\{ \frac{1}{T^{1/(1+\alpha)} L\left(T^{1/(1+\alpha)}\right)^{1/(1+\alpha)}} X^*(Tt) \right\} \xrightarrow{d} \left\{ S_{1+\alpha}(t) \right\},$$

where $\{S_{1+\alpha}\}$ *is an* $(1+\alpha)$*-stable Lévy process.*

Note: $1+\alpha$ dominates β. Note also that $S_{1+\alpha}(t)$ has infinite variance since $1+\alpha < 2$ and has independent increments.

Dependent stable process

Theorem

Suppose that

- $\mathbb{E}X(t) = 0, \sigma^2 = \mathrm{Var}X(t) < \infty$
- π *involves* $\alpha \in (0, 1)$ *and some slowly varying function* L.
- *There is no Gaussian component*
- *The conditions on* M *hold with* $\beta > 0$,
- $$0 < 1 + \alpha < \beta < 2.$$

Then as $T \to \infty$,

$$\left\{ \frac{1}{T^{1-\alpha/\beta} L(T)^{1/\beta}} X^*(Tt) \right\} \xrightarrow{d} \{Z_{\alpha,\beta}(t)\},$$

where $\{Z_{\alpha,\beta}\}$ *is* β*-stable* $(1 - \alpha/\beta)$*-self-similar process with stationary increments given by the stochastic integral representation*

$$Z_{\alpha,\beta}(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} (\mathfrak{f}(x, t - s) - \mathfrak{f}(x, -s))\, S_\beta(dx, ds), \tag{8}$$

where

$$\mathfrak{f}(x, u) = x^{-1}(1 - e^{-xu}) \mathbf{1}_{x>0} \mathbf{1}_{u>0},$$

and where S_β *is a* β*-stable random measure on* $\mathbb{R}_+ \times \mathbb{R}$ *with control measure* $\alpha x^\alpha dxds$.

Notes about the limit

- The limit is

$$Z_{\alpha,\beta}(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\mathfrak{f}(x, t-s) - \mathfrak{f}(x, -s)\right) S_\beta(dx, ds). \tag{9}$$

- This process was first obtained by Pilipauskaité and Surgailis (2010), *Advances in Applied Probability* 42.2 (2010) 509-527, in their study of the aggregation of random AR(1) processes.
- The process has stationary but dependent increments.
- The process is self-similar with $H = (1 - \alpha/\beta) \in (1/2, 1)$.
- It is a stable self-similar *mixed* moving average (because of the x variable)
- $$\mathfrak{f}(x, t-s) - \mathfrak{f}(x, -s) = \begin{cases} x^{-1} e^{-xs}(1 - e^{-xt}), & \text{if } s < 0 \\ x^{-1}(1 - e^{-x(t-s)}), & \text{if } 0 \le s \le t \\ 0, & \text{otherwise.} \end{cases}$$

Where is the intermittency best seen?

Perhaps in the following theorem where the limit is FBM, but

- X^* is not intermittent if it is purely Gaussian and
- X^* is intermittent if it has also a pure jump component.

Theorem

Suppose that the supOU process $X(t)$ is defined using a Lévy process which is

purely Gaussian

or having

also a pure jump component

with $\mathbb{E}X(t) = 0$, $\sigma^2 = \mathrm{Var}X(t) < \infty$ and $\alpha \in (0,1)$ with some slowly varying function L. Then in both cases, as $T \to \infty$

$$\left\{ \frac{1}{T^{1-\alpha/2} L(T)^{1/2}} X^*(Tt) \right\} \xrightarrow{d} \{\widetilde{\sigma} B_H(t)\},$$

where $\{B_H(t)\}$ is fractional Brownian motion with $H = 1 - \alpha/2 \in (1/2, 1)$.

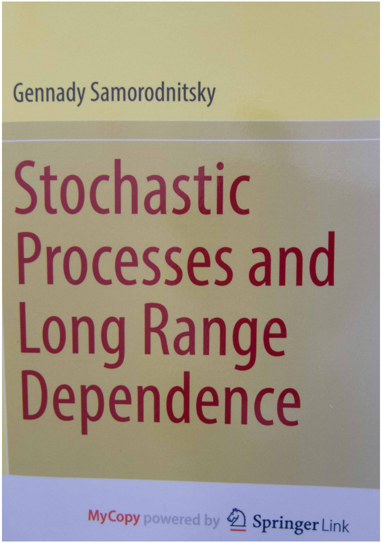
$X^*(t)$ is intermittent only in the second case.

Conclusion

Using supOU processes, we showed that:

Limit theorems and intermittency can occur jointly;

Intermittency involves an unusual behavior of the moments.



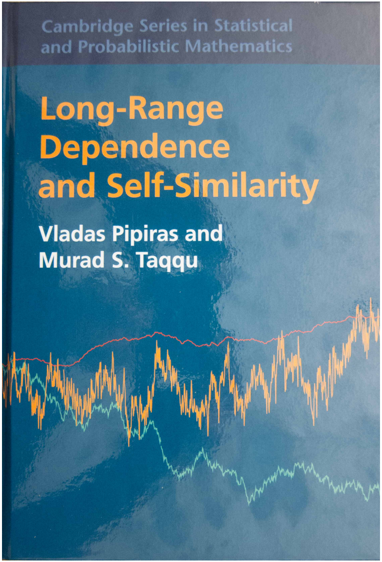
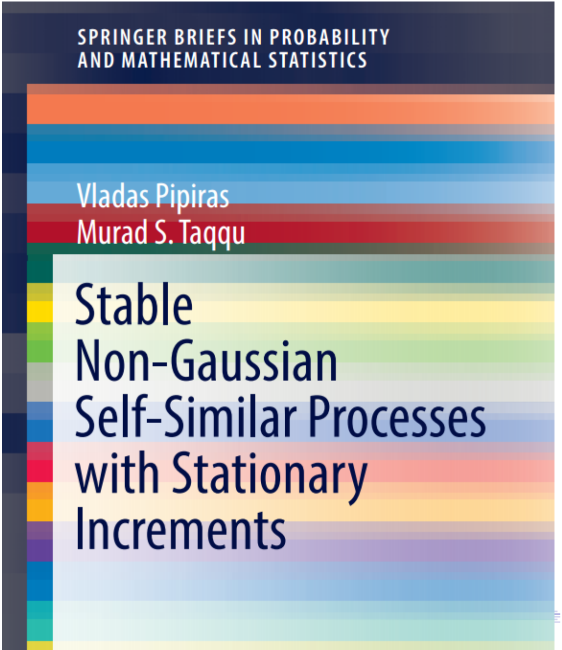


Figure : Two new books on LRD


SPRINGER BRIEFS IN PROBABILITY AND MATHEMATICAL STATISTICS
Vladas Pipiras
Murad S. Taqqu
Stable Non-Gaussian Self-Similar Processes with Stationary Increments

THANK YOU

Limit chart

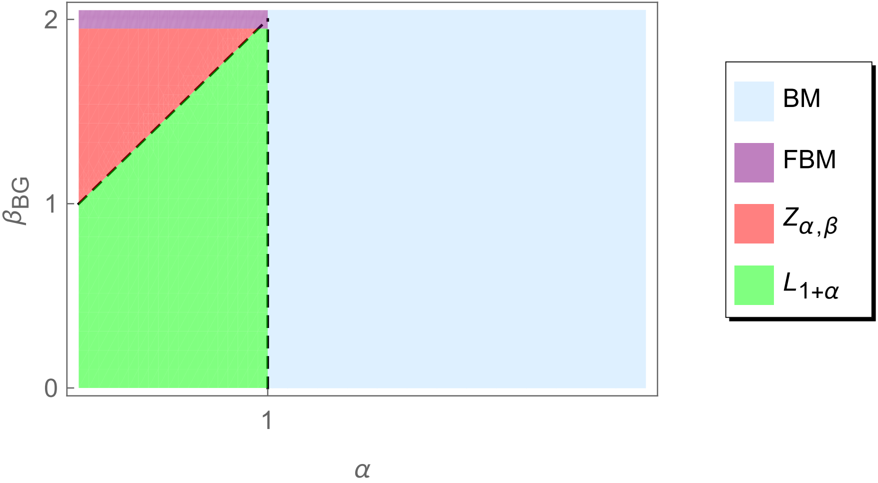


Figure :

Intermittency lemma with cumulants

Theorem

Let $X(t)$, $t > 0$ be a supOU process such that

- $\pi(0, x) \equiv L(x^{-1})x^{\alpha}$ *as* $x \to 0$, $\alpha > 0$.
- *The cumulant function of* $X(t)$ *is analytic around the origin.*

Then for any $q > \alpha + 1$ *and* $\kappa_X^{(q)} \neq 0$, *we have*

$$\sigma_{X^*}(q) = q - \alpha.$$

Difference in the assumptions:

Cumulant lemma: $\alpha > 0$, $q > \alpha + 1$, $\kappa_X^{(q)} \neq 0$.

Conclusion involves $\sigma_{X^*}(q) = \lim_{t\to\infty} \frac{\log |\kappa_{X^*}^{(q)}|}{\log t}$: $\sigma_{X^*}(q) = q - \alpha$

Moment theorem: assume $X(t)$ non-Gaussian, $\alpha > 0$, $q \geq q^*$, where q^* is the smallest even integer greater than 2α, $\mathbb{E}X(t) = 0$, $\mathbb{E}X(t)^2 \neq 0$.

Conclusion involves $\tau_{X^*}(q) = \lim_{t\to\infty} \frac{\log \mathbb{E}|X^*(t)|^q}{\log t}$: $\tau_{X^*}(q) = q - \alpha$

Sketch of proof of the FBM theorem

Case $0 < 1+\alpha < \beta < 2$. Infinite variance since $\beta < 2$.

Decompose $X(t) = X_1(t) + X_2(t)$ where $X_1(t)$ has the Gaussian component and $X_2(t)$ has the pure jump component.

$$\left\{ \frac{1}{T^{1-\alpha/2} L(T)^{1/2}} X_1^*(Tt) \right\} \xrightarrow{d} \{\widetilde{\sigma} B_H(t)\},$$

$$\left\{ \frac{1}{T^{1-\alpha/\beta} L(T)^{1/\beta}} X_2^*(Tt) \right\} \xrightarrow{d} \{Z_{\alpha,\beta}(t)\},$$

But

$$\begin{aligned} \frac{1}{T^{1-\alpha/2} L(T)^{1/2}} X_2^*(Tt) &= \frac{1}{T^{1-\alpha/2} L(T)^{1/2}} \frac{T^{1-\alpha/\beta} L(T)^{1/\beta}}{T^{1-\alpha/\beta} L(T)^{1/\beta}} X_2^*(Tt) \\ &\overset{d}{\approx} \frac{T^{1-\alpha/\beta} L(T)^{1/\beta}}{T^{1-\alpha/2} L(T)^{1/2}} Z_{\alpha,\beta}(t) \\ &= T^{-\alpha(1/\beta - 1/2)} L(T)^{(1/\beta-1/2)} Z_{\alpha,\beta}(t) \to 0 \end{aligned}$$

since $1/\beta - 1/2 = \frac{2-\beta}{2\beta} > 0$. Hence only $X_1(t)$ contributes to the limit.