Limit theorems for the realised covariation of a bivariate Brownian semistationary process

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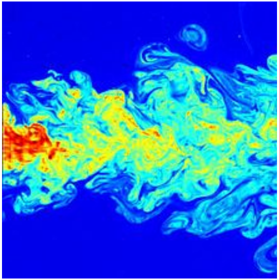
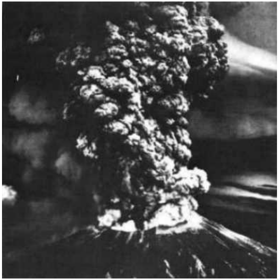
Motivation: Ambit stochastics

- Name for the theory and applications of ambit fields and ambit processes
- **Probabilistic framework for spatio-temporal modelling**
- Introduced by O. E. Barndorff-Nielsen and J. Schmiegel in the context of modelling turbulence in physics.

- In this talk, we focus on the null-spatial case of an ambit field:

A Brownian semistationary process.

The Brownian semistationary (BSS) process

- The BSS process in its most basic form can be written as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \, dW_s,$$

for a deterministic kernel function g, a stochastic volatility process σ and a Brownian motion W.

- Areas of application:
  - Turbulence (Barndorff-Nielsen & Schmiegel (2009)),
  - (energy) finance (Barndorff-Nielsen, Benth, V. (2013)),
  - (rough) volatility (Bennedsen, Lunde, Pakkanen (2017+)).
- Recent interest in inference on σ (and related quantities) in particular in the case when Y is NOT a semimartingale.
- Related work: Power variation for Gaussian, BSS and LSS processes: Barndorff-Nielsen, Corcuera, Podolskij, and Woerner (2009), Barndorff-Nielsen, Corcuera, Podolskij (2011, 2013), Corcuera (2012), Corcuera, Hedevang, Pakkanen, and Podolskij (2013), Basse-O'Connor, Heinrich and Podolskij (2017), Basse-O'Connor,Lachiéze-Rey and Podolskij (2017), Basse-O'Connor and Podolskij (2017) etc.

Aim of the project

- Consider a bivariate semimartingale $\mathbf{Y} = (Y^{(1)}, Y^{(2)})^\top$ sampled at high frequency with increments given by $\Delta_i^n Y^{(j)} = Y_{i\Delta_n}^{(j)} - Y_{(i-1)\Delta_n}^{(j)}$, for $j \in \{1, 2\}$ and for $\Delta_n = n^{-1}$ with $n \in \mathbb{N}$.
- For $t \geq 0$, we call

$$\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}$$

the realised covariation.
- It is well-known that the quadratic covariation denoted by $[Y^{(1)}, Y^{(2)}]$ exists and that

$$\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \stackrel{u.c.p.}{\to} [Y^{(1)}, Y^{(2)}]_t, \text{ as } n \to \infty.$$

- Aim: Derive the asymptotic properties of (the possibly scaled) **realised covariation** for a **bivariate BSS** process $\mathbf{Y}$ outside the semimartingale framework.

General remarks

- Focus on the non-semimartingale case throughout the study.
- Aims: Derive both a weak law of large numbers and a central limit theorem for (scaled) realised covariation.
- Method of proofs:
  - Focus on the bivariate Gaussian core first, i.e. a bivariate Brownian semistationary process without stochastic volatility.
  - Derive all results in the absence of stochastic volatility.
  - Extend all results to the general case with stochastic volatility using Bernstein's blocking technique, where the volatility is "frozen" on a coarser time grid.
  - The weak law of large numbers can be proven using classical techniques for convergence of measures.
  - For the central limit theorem we invoke the powerful fourth moment theorem by Nualart and Peccati (2005).

The setting

- Consider filtered, complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a finite time horizon $[0, T]$ for some $T > 0$.
- Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ supports two independent $\mathcal{F}_t$-Brownian measures $W^{(1)}, W^{(2)}$ on $\mathbb{R}$.

Definition 1 (Brownian measure)

An $\mathcal{F}_t$-adapted Brownian measure $W\colon \Omega \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a Gaussian stochastic measure such that, if $A \in \mathcal{B}(\mathbb{R})$ with $\mathbb{E}[(W(A))^2] < \infty$, then $W(A) \sim N(0, Leb(A))$, where Leb is the Lebesgue measure. Moreover, if $A \subseteq [t, +\infty)$, then $W(A)$ is independent of $\mathcal{F}_t$.

Definition 2 (The Gaussian core)

Consider two Brownian measures $W^{(1)}$ and $W^{(2)}$ adapted to $\mathcal{F}_t$ with $dW_t^{(1)} dW_t^{(2)} = \rho dt$, for $\rho \in [-1, 1]$. Further take two nonnegative deterministic functions $g^{(1)}, g^{(2)} \in L^2((0, \infty))$ which are continuous on $\mathbb{R} \setminus \{0\}$. Define, for $j \in \{1, 2\}$,

$$G_t^{(j)} := \int_{-\infty}^{t} g^{(j)}(t-s)\, dW_s^{(j)}.$$

Then the vector process $(\mathbf{G}_t)_{t\geq 0} = (G_t^{(1)}, G_t^{(2)})_{t\geq 0}^{\top}$ is called the (bivariate) Gaussian core.

The bivariate Brownian semistationary process

Definition 3 (The bivariate Brownian semistationary process)

Consider two Brownian measures $W^{(1)}$ and $W^{(2)}$ adapted to $\mathcal{F}_t$ with $dW_t^{(1)} dW_t^{(2)} = \rho dt$, for $\rho \in [-1, 1]$. Further take two nonnegative deterministic functions $g^{(1)}, g^{(2)} \in L^2((0, \infty))$ which are continuous on $\mathbb{R} \setminus \{0\}$. Let further $\sigma^{(1)}, \sigma^{(2)}$ be càdlàg , $\mathcal{F}_t$-adapted stochastic processes and assume that for $j \in \{1, 2\}$, and for all $t \in [0, T]$: $\int_{-\infty}^{t} g^{(j)2}(t - s)\sigma_s^{(j)2} \, ds < \infty$. Define, for $j \in \{1, 2\}$,

$$Y_t^{(j)} := \int_{-\infty}^{t} g^{(j)}(t - s)\sigma_s^{(j)} \, dW_s^{(j)}.$$

Then the vector process $(\mathbf{Y}_t)_{t \geq 0} = (Y_t^{(1)}, Y_t^{(2)})_{t \geq 0}^{\top}$ is called a bivariate Brownian semistationary process.

Technical assumptions

Assumption 1

For $j \in \{1,2\}$, we assume that $g^{(j)} : \mathbb{R} \to \mathbb{R}^+$ are nonnegative functions and continuous, except possibly at $x = 0$. Also, $g^{(j)}(x) = 0$ for $x < 0$ and $g^{(j)} \in L^2((0,+\infty))$. We further ask that $g^{(j)}$ be differentiable everywhere with derivative $\left(g^{(j)}\right)' \in L^2((b^{(j)},\infty))$ for some $b^{(j)} > 0$ and $\left((g^{(j)})'\right)^2$ non-increasing in $[b^{(j)},\infty)$.

- Set $b = \max\{b^{(1)}, b^{(2)}\}$.
- It is important to note that we are not assuming that $\left(g^{(j)}\right)' \in L^2((0,\infty))$ in order to exclude the semimartingale case. In particular, we must have that, for all $\varepsilon > 0$, $\sup_{x\in(0,\varepsilon)} \left(g^{(j)}\right)'(x) = \infty$.

Technical assumptions cont'd

For $i, j \in \{1, 2\}$, we write $\rho_{i,j} = \rho$ for $i \neq j$ and $\rho_{i,j} = 1$ for $i = j$. Also, for $i, j \in \{1, 2\}$ set: $\bar{R}^{(i,j)}(t) := \mathbb{E}\left[\left(G_t^{(j)} - G_0^{(i)}\right)^2\right]$.

Assumption 2

For all $t \in (0, T)$, there exist slowly varying functions $L_0^{(i,j)}(t)$ and $L_2^{(i,j)}(t)$ which are continuous on $(0, \infty)$ such that

$$\bar{R}^{(i,j)}(t) = C_{i,j} + \rho_{i,j} t^{\delta^{(i)}+\delta^{(j)}+1} L_0^{(i,j)}(t), \quad \text{for } i, j \in \{1, 2\}, \tag{1}$$

and

$$\frac{1}{2}(\bar{R}^{(i,j)})''(t) = \rho_{i,j} t^{\delta^{(i)}+\delta^{(j)}-1} L_2^{(i,j)}(t), \quad \text{for } i, j \in \{1, 2\},$$

where $\delta^{(1)}, \delta^{(2)} \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$.

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Technical assumptions cont'd

Assumption 3 (Assumption 2 cont'd)

Also, if we denote $\tilde{L}_0^{(i,j)}(t) := \sqrt{L_0^{(i,i)}(t) L_0^{(j,j)}(t)}$*, we ask that the functions* $L_0^{(i,j)}(t)$ *and* $L_2^{(i,j)}(t)$ *are such that, for all* $\lambda > 0$*, there exists a* $H^{(i,j)} \in \mathbb{R}$ *such that:*

$$\lim_{t \to 0+} \frac{L_0^{(i,j)}(\lambda t)}{\tilde{L}_0^{(i,j)}(t)} = H^{(i,j)} < \infty, \tag{2}$$

and that there exists $b \in (0, 1)$*, such that:*

$$\limsup_{x \to 0^+} \sup_{y \in (x, x^b)} \left| \frac{L_2^{(i,j)}(y)}{\tilde{L}_0^{(i,j)}(x)} \right| < \infty. \tag{3}$$

In this situation, the restriction $\delta^{(j)} \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ ensures that the process leaves the semimartingale class.

Example of a relevant kernel function

Example 4

The Gamma kernel satisfies all our assumptions:

$$g^{(j)}(x) = x^{\delta^{(j)}} e^{-\lambda^{(j)} x},$$

for $x \geq 0$, $\lambda^{(j)} > 0$ and $\delta^{(j)} \in (-0.5, 0.5) \setminus \{0\}$ and $j \in \{1, 2\}$.

The scaling factor

- For $j \in \{1, 2\}$, set

$$\tau_n^{(j)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n G^{(j)}\right)^2\right]}$$
$$= \sqrt{\int_0^\infty \left(g^{(j)}(s + \Delta_n) - g^{(j)}(s)\right)^2 ds + \int_0^{\Delta_n} \left(g^{(j)}(s)\right)^2 ds}.$$

- The scaled realised covariation of the Gaussian core is given by

$$\sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}}.$$

- Let us now derive the central limit theorem!

Weak Convergence of the Gaussian Core

Theorem 5 (Weak Convergence of the Gaussian Core)

Assume that Assumptions 1 and 2 hold with $\delta^{(1)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$*. Then we obtain:*

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n G^{(2)}}{\tau_n^{(2)}} \right] \right) \right)_{t \in [0,T]} \Rightarrow \left(\sqrt{\beta} B_t \right)_{t \in [0,T]}, \tag{4}$$

where B_t *is a Brownian motion independent of the processes* $G^{(1)}$, $G^{(2)}$, β *is a known constant and the convergence is in the Skorokhod space* $\mathcal{D}[0, T]$ *equipped with the Skorokhod topology.*

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Even more technical assumptions...

Assumption 4

We require that, for $k \in \{1,2\}$, the quantity:

$$\frac{\sqrt{\mathbb{E}\left[\left(\int_{-\infty}^{(i-1)\Delta_n} \Delta g^{(k)} \sigma_s^{(k)}\, dW_s^{(k)}\right)^2\right]}}{\tau_n^{(k)}} = \frac{\sqrt{\int_0^{\infty} \left(g^{(k)}(s+\Delta_n) - g^{(k)}(s)\right)^2 \mathbb{E}\left[\left(\sigma_{(i-1)\Delta_n - s}^{(k)}\right)^2\right] ds}}{\tau_n^{(k)}}$$

is uniformly bounded in $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.

Technical assumptions cont'd

Assumption 5

The stochastic volatility process $\sigma^{(1)}$ (resp. $\sigma^{(2)}$) has $\alpha^{(1)}$-Hölder (resp. $\alpha^{(2)}$) continuous sample paths, for $\alpha^{(1)} \in \left(\frac{1}{2}, 1\right)$. Furthermore, both the kernel functions $g^{(1)}$ and $g^{(2)}$ satisfy the following property: For $j \in \{1, 2\}$, write:

$$\pi_n^{(j)}(A) := \frac{\int_A \left(g^{(j)}(x + \Delta_n) - g^{(j)}(x)\right)^2 ds}{\int_0^\infty \left(g^{(j)}(x + \Delta_n) - g^{(j)}(x)\right)^2 ds}$$

and note that $\pi_n^{(j)}$ are probability measures. We ask that there exists a constant $\lambda < -1$ such that for any $\varepsilon_n = O(n^{-\kappa})$, it holds that:

$$\pi_n^{(j)}\left((\varepsilon_n, \infty)\right) = O\left(n^{\lambda(1-\kappa)}\right).$$

Central limit theorem

Theorem 6 (Central limit theorem)

Let $\mathcal{G}$ be the sigma algebra generated by the Gaussian core $\mathbf{G}$*, and let $\sigma^{(1)}$ and $\sigma^{(2)}$ be $\mathcal{G}-$measurable. For the bivariate $\mathcal{BSS}$ process, provided that Assumptions 1-4 are satisfied with $\delta^{(1)}, \delta^{(2)} \in (-\frac{1}{2}, \frac{1}{4}) \setminus \{0\}$, the following $\mathcal{G}$-stable convergence holds:*

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(1)}}{\tau_n^{(1)}} \frac{\Delta_i^n Y^{(2)}}{\tau_n^{(2)}} - \sqrt{n}\,\mathbb{E}\left[\frac{\Delta_1^n G^{(1)}}{\tau_n^{(1)}} \frac{\Delta_1^n G^{(2)}}{\tau_n^{(2)}} \right] \int_0^t \sigma_s^{(1)} \sigma_s^{(2)}\, ds \right)_{t \in [0,T]} \overset{st.}{\underset{n \to \infty}{\Longrightarrow}} \left(\sqrt{\beta} \int_0^t \sigma_s^{(1)} \sigma_s^{(2)}\, dB_s \right)_{t \in [0,T]}, \quad (5)$$

in the Skorokhod space $\mathcal{D}[0, T]$, where β is a known constant. Also, B is Brownian motion, independent of $\mathcal{F}$ and defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

A weak law of large numbers

- We set

$$c(x) := \int_0^x g^{(1)}(s) g^{(2)}(s)\, ds + \int_0^\infty \left(g^{(1)}(s+x) - g^{(1)}(s) \right) \left(g^{(2)}(s+x) - g^{(2)}(s) \right) ds.$$

Proposition 1

Assume that the conditions of the previous theorem hold. Then

$$\frac{\Delta_n}{c(\Delta_n)} \sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \xrightarrow{\mathbb{P}} \rho \int_0^t \sigma_s^{(1)} \sigma_s^{(2)}\, ds, \qquad \text{as } n \to \infty.$$

Summary and outlook

- We derived a central limit theorem for the scaled realised covariation of a bivariate Brownian semistationary process.
- A weak law of large numbers is implied under the same assumptions.
- We have also extended the derivations of the weak law of large numbers to another non-semimartingale scenario within the BSS class.
- Detailed results and derivations are available in
  - Granelli & Veraart (2017), "A central limit theorem for the realised covariation of a bivariate Brownian semistationary process", eprint arXiv:1707.08507
  - Granelli & Veraart (2017), "A weak law of large numbers for estimating the correlation in bivariate Brownian semistationary processes", eprint arXiv:1707.08505
- Outlook: Asymptotic theory for general multivariate BSS processes; feasible inference; relative co-volatility; realised betas etc. [Ongoing work with Riccardo Passeggeri (Imperial College London)].