Some Remarks on Scaling and Universality in Turbulence

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The relation between universality and temporal and spatial scaling of structure-functions and moments of the coarse-grained energy dissipation in turbulence is studied for flows where the Taylor Frozen Flow Hypothesis holds exactly. To account for observed deviations from a strict scaling law, we conclude that non-constant scaling exponents depend on the mean flow U. Furthermore, spatial and temporal scaling exponents are identical if and only if they are constant.

KEYWORDS: Turbulence; Scaling; Taylor Frozen Flow Hypothesis.

I. INTRODUCTION

Turbulence is one of the most challenging subjects of classical physics. The underlying hydrodynamical equation, the Navier-Stokes equation for an incompressible flow (Navier (1823))

$$\partial_t \vec{v}(\vec{r},t) + \vec{v}(\vec{r},t) \cdot \vec{\nabla} \vec{v}(\vec{r},t) = -\vec{\nabla} p(\vec{r},t) + \nu \vec{\nabla}^2 \vec{v}(\vec{r},t) \tag{1}$$

has been known for nearly two centuries, but only a very small part of its dynamical content is revealed so far (Frisch (1995)). Here $\vec{v}(\vec{r},t)$ denotes the velocity of the flow at spatial position $\vec{r}$ and time t. ν is the kinematic viscosity, p the true pressure divided by the constant density ρ_0 and $\vec{\nabla}$, $\vec{\nabla}^2$ and ∂_t are the gradient, Laplace operator and partial derivative with respect to time t, respectively. This equation has to be supplemented by the incompressibility condition $\vec{\nabla} \cdot \vec{v}(\vec{r},t) = 0$ and boundary and initial conditions. A simple and far-reaching aspect of this non-linear and non-local equation is the scaling symmetry, i.e. the Navier-Stokes equation is invariant under a change of $\vec{r} \rightarrow \lambda \vec{r}$, $\vec{v} \rightarrow \lambda \vec{v}$ and $\nu \rightarrow \lambda^2 \nu$ for all $\lambda \in \mathbb{R}_{>0}$. Stated otherwise, it reflects the similarity principle for an incompressible flow (Monin and Yaglom (1971)). For a given geometrical shape of the boundaries, the Reynolds number

$$\mathrm{Re} = \frac{LU}{\nu}$$

is the only control parameter of the flow. Here L and U denote, respectively, a characteristic scale and a characteristic velocity of the flow. For a steady and homogeneous flow, U usually is the constant mean velocity. L is associated with the size of the flow (like the diameter of a pipe) or with the size of an obstacle generating disturbances.. Increasing the Reynolds number changes the character of the flow from a laminar one into a turbulent state where the velocity field is highly irregular and intermittent and only permits a statistical description of the flow (Frisch (1995)).

In this paper the discussion is restricted to the simplest case of a homogeneous and steady flow with respect to a one-dimensional cut of the velocity field along the direction of the constant mean flow $\vec{U}$ (assumed to exist). Homogeneity and steadiness means that the law of $(v(x,t))_{x,t}$ is the same as that of $(v(x+\Delta x, t+\Delta t))_{x,t}$ for arbitrary Δx and Δt. Here x

denotes the coordinate in direction of the constant mean flow $\vec{U}$ and v is one component of $\vec{v}$.

The most commonly studied statistical quantities in this one-dimensional set-up are the longitudinal spatial structure-functions

$$\mathrm{S}_s^{(p)}(l) \equiv \mathrm{E}\left\{(v(x+l,t)-v(x,t))^p\right\}, \tag{2}$$

where v is the component of the velocity in the direction of the mean flow $\vec{U}$. The subscript s emphasises the spatial character of definition (2), since it only involves a spatial distance l. It is important to note that due to homogeneity and steadiness, the left hand side of (2) does not depend on x or t.

Another set of basic statistical quantities in turbulence are the moments of the spatially coarse grained energy dissipation

$$\overline{\epsilon}_s^{(p)}(l) \equiv \mathrm{E}\left\{\left(\frac{1}{l}\int_{x-l/2}^{x+l/2} \epsilon(\sigma,t)\mathrm{d}\sigma\right)^p\right\} \tag{3}$$

where

$$\epsilon(x,t) = 15\nu\left(\partial_x v(x,t)\right)^2 \tag{4}$$

is the so-called surrogate energy dissipation. The true energy dissipation as a measure for the rate of the loss of kinetic energy in a three-dimensional turbulent flow includes all derivatives $\partial_i v_j(\vec{r},t)$, $i,j = x,y,z$, $\vec{v} = (v_x, v_y, v_z)$ (Frisch (1995), Monin and Yaglom (1971)). In the sequel the term energy dissipation always refers to the surrogate definition (4). As for spatial structure functions $S_s^{(p)}(l)$ the subscript s in (3) indicates spatial coarse graining. Again, due to homogeneity and steadiness of the flow, the left hand side of (3) does not depend on the location x of the coarse graining domain of length l and is independent of time t.

A basic hypothesis in turbulence states that for a homogeneous and steady flow with $\mathrm{Re} \to \infty$, the structure functions $S_s^{(p)}(l)$ and the moments of the coarse grained energy dissipation $\overline{\epsilon}_s^{(p)}(l)$ fulfill multifractal scaling relations

$$\mathrm{S}_s^{(p)}(l) = c_s(p,U)\,l^{\xi_s(p)} \tag{5}$$

and

$$\overline{\epsilon}_s^{(p)}(l) = d_s(p,U)\,l^{\tau_s(p)}, \tag{6}$$

where l is within the so-called inertial range (Frisch (1995), Sreenivasan and Antonia (1997)). A working definition is the range where (5) holds for $p=3$, the famous Kolmogorov 4/5-th law (Kolmogorov (1941a,b), Frisch (1995)). The constant prefactors $c_s(p,U)$ and $d_s(p,U)$ are assumed to be independent of l and cover all flow specific features. The scaling exponents $\xi_s(p)$ and $\tau_s(p)$ are supposed to be universal in the sense that they are flow independent. Universality in general refers to features of a turbulent flow that are independent of the experimental set-up, in particular independent of the mean flow U. The appearence of an argument U as for c_s and d_s in (7) and (8) indicates the non-universal character of these quantities.

The term multifractal scaling refers to the fact that the scaling exponents $\xi_s(p)$ and $\tau_s(p)$ are nonlinear functions of the order p. For a more precise definition of multifractality and its connection to scaling exponents in turbulence, the reader is refered to (Sreenivasan (1991) and numerous references therein).

Much effort has been devoted towards verifying the scaling relations (5) and (6) (see Sreenivasan (1997) for an overview). Most of these investigations are based on the measurement of the velocity v at a fixed position in space and as a function of time. These time series are then transformed into a spatial resolution of the velocity field using the so-called Taylor Frozen Flow Hypothesis (TFFH) (Taylor (1938)). TFFH states that in the presence of a constant mean flow U and under the assumption that the fluctuations of the velocity are small compared to the constant mean flow, turbulent structures are advected by the mean flow. Accordingly, the time series may be interpreted as spatial recordings by replacing $t \to x = Ut$. In its full consequence TFFH implies that the law of $(v(x_0, t_0 + t))_t$, (x_0, t_0) fixed, is the same as the law of $(v(x_0 - Ut, t_0))_t$, (x_0, t_0) fixed. The important prerequisites are that there is a constant mean flow U and that turbulent fluctuations are small compared to the mean flow U. The resulting spatial resolution is then along the direction x of the mean flow U.

In situations where TFFH is violated one has to use other techniques, like RELIEF (laser induced electronic fluorescence) (Noullez et al (1997)), PHANTOMM (photo-activated non-intrusive tracking of molecular motion) (Lempert et al (1995), Harris et al (1996)) and PIV (particle image velocimetry) (Wereley and Lueptow (1998)) for directly observing the spatial distribution of the velocity field. But these methods are either not applicable to the measurement of the longitudinal velocity component along the direction of the mean flow or only give a poor resolution. In case of the absence of a constant mean flow and when turbulent fluctuations are large, there are possible corrections to TFFH involving a locally averaged advection velocity (Pinton and Labbé (1994)).

Here we only consider situations where a constant mean flow U exists, TFFH holds exactly and the flow is homogeneous and steady. In this case it is easy to show that the spatial scaling relations (5) and (6) are equivalent to their temporal counterparts (denoted by the subscript T)

$$\mathrm{S}_T^{(p)}(l) \equiv \mathrm{E}\left\{(v(x, t-l) - v(x,t))^p\right\} = c_T(p, U) l^{\xi_T(p)} \tag{7}$$

and

$$\bar{\epsilon}_T^{(p)}(l) \equiv \mathrm{E}\left\{\left(\frac{1}{l}\int_{t-l/2}^{t+l/2} \epsilon(x,s)\mathrm{d}s\right)^p\right\} = d_T(p,U) l^{\tau_T(p)}. \tag{8}$$

Again, the constants $c_T(p, U)$ and $d_T(p, U)$ contain all flow specific features. The exponents τ_T and ξ_T are universal, i.e. do not depend on the mean flow U.

The focus of this paper is the discussion of deviations from universal scaling (5), (6) and (7), (8) in relation to TFFH. In Section II, a factorisation Ansatz for a generalisation of scaling relations is proposed to account for experimentally observed deviations from scaling. In particular, we investigate constraints that are imposed by TFFH on possible deviations from scaling, reflected in non-constant and non-universal scaling exponents $\tau(p, U, l)$ and $\xi(p, U, l)$. Section III discusses implications of TFFH on the functional form of these deviations with respect to universality. The main result is that these deviations are necessarily non-universal, i.e. depend on the mean flow U. Section IV concludes.

II. SCALING AND UNIVERSAL FACTORISATION

Experiments on high Reynolds number turbulence indicate that the scaling relations (5), (6) and (7), (8) might only be approximations (Sreenivasan and Antonia (1997), Schmiegel et al (2002))). Even for very high Reynolds numbers (up to order 10^8 in atmospheric turbulence), the scaling behaviour of structure functions and the moments of the coarse grained energy dissipation is not constant within the inertial range. The obvious way to proceed is then to replace the constant and universal scaling exponents $\xi(p)$ and $\tau(p)$ with in general slowly varying functions $\xi(p,l)$ and $\tau(p,l)$ within a range of scales l where this variation is supposed to be negligible. It is not clear, whether this procedure leads to universality (in the sense of idependence from U) of the scaling exponents. As will be shown in the next section, TFFH implies that the non-universality of the functions $\xi(p,U,l)$ and $\tau(p,U,l)$ is compulsory.

Second, the range of scales where scaling approximately holds only covers part of the range of scales that are characteristic for a turbulent flow.

This raises the question whether a characterisation of universality in turbulence via scaling laws for $S^{(p)}(l)$ and $\overline{\epsilon}^{(p)}(l)$ is appropriate, since these quantities restrict universality considerations to a more or less small part of accessible scales. Barndorff-Nielsen et al (2004) propose as an alternative a universal description of the probability density of velocity increments in terms of the Normal Inverse Gaussian distribution that covers all accessible scales from the finest resolution up to integral scales.

Furthermore, even if scaling relations are the right tool for detecting universal features of a turbulent flow, it is not clear that structure functions and moments of the coarse grained energy dissipation are well suited quantities for this purpose. In fact, recent work on the statistics of the energy dissipation (Barndorff-Nielsen et al (2003), Barndorff-Nielsen and Schmiegel (2003), Schmiegel et al (2004)) indicates that two-point expectations $\mathrm{E}\{\epsilon(x,t)^{n_1}\epsilon(x,t)^{n_2}\}$, $n_1,n_2\in\mathbb{N}_{>0}$, display a more strict scaling behaviour at a range of scales that is appreciably larger than that of $\overline{\epsilon}^{(p)}$.

Here we only adress the first problem and ask about a generalisation of (5), (6) and (7), (8) to account for more general situations than pure power-laws. The requirement for these generalisations to be compatible with TFFH then links to the kind of universality that can be expected in these generalisations.

An obvious generalisation consists in keeping the factorization into universal and non-universal terms that is inherent to (5), (6) and (7), (8) but replacing the universal power law behaviour by arbitrary, universal functions $f_s(p,l)$, $g_s(p,l)$ and $f_T(p,l)$, $g_T(p,l)$

$$\mathrm{S}_s^{(p)}(l) = c_s(p,U)f_s(p,l), \tag{9}$$

$$\mathrm{S}_T^{(p)}(l) = c_T(p,U)f_T(p,l) \tag{10}$$

and

$$\overline{\epsilon}_s^{(p)}(l) = d_s(p,U)g_s(p,l). \tag{11}$$

$$\overline{\epsilon}_T^{(p)}(l) = d_T(p,U)g_T(p,l). \tag{12}$$

The constants c_s, d_s and c_T, d_T are assumed to contain all flow-specific features, indicated by their dependence on U. It is to note that it is not assumed that the universal functions f_T and g_T are the same functions as f_s and g_s. Thus (9), (10) and (11), (12) are a rather general Ansatz to account for deviations from the strict scaling relations (5), (6) and (7), (8). However, as will be shown soon, it is not general enough once TFFH is invoked.

In what follows the term spatial factorisation refers to (9) and (11) and temporal factorisation refers to (10) and (12). Spatial scaling and temporal scaling refers to situations where $f_s(l,p)$, $g_s(l,p)$ and $f_T(l,p)$, $g_T(l,p)$ are power-laws, respectively. Universality will always refer to independence of the mean flow U.

The aim of the analysis in Section III is to show that in the presence of the validity of TFFH, spatial and temporal factorisation is equivalent to spatial and temporal scaling. Stated otherwise, if TFFH holds and if the structure-functions and moments of the coarse grained energy dissipation factorize into a flow dependent and flow independent part, than the scaling rules (5), (6) and (7), (8) are compulsory.

This result has far-reaching impact on the problem, whether deviations from scaling can be universal. It also implies that, under the assumption of TFFH and spatial and temporal factorisation, only constant scaling exponents τ and ξ are universal.

III. TFFH AND UNIVERSALITY

The framework to describe the statistics of the velocity field and the coarse grained energy dissipation is assumed to be defined by TFFH (including the existence of a constant mean flow U) and equations (9), (10) and (11), (12), together with homogeneity and steadiness of the flow. Here we restrict the analysis to structure functions. The results are equally valid for moments of the coarse grained energy dissipation.

TFFH states that the law of $(v(x_0, t_0 + t))_t$, $(x_0, t_0$ fixed) is the same as that of $(v(x_0 - Ut, t_0))_t$, $(x_0, t_0$ fixed). Due to homogeneity and steadiness we set $x_0 = 0$ and $t_0 = 0$, for convenience. This immediatedly implies for the temporal and spatial structure functions that

$$S_T^{(p)}(l) = S_s^{(p)}(Ul). \tag{13}$$

Inserting (9) and (10) in (13) gives

$$c_T(p, U) f_T(p, l) = c_s(p, U) f_s(p, Ul). \tag{14}$$

Then we have, since $f_T(p, l)/f_s(p, Ul)$ is independent of l,

$$f_T(p, l) = f_s(p, l) = l^{\xi(p)} \tag{15}$$

and

$$c_T(p, U) = c_s(p, U) U^{\xi(p)}. \tag{16}$$

The universal exponent $\xi(p)$ in (14) is the same for spatial and temporal structure functions. The same line of arguments also holds for the moments of the coarse grained energy dissipation. Thus temporal and spatial scaling with identical scaling exponents $\tau_s(p) = \tau_T(p) = \tau(p)$ and $\xi_s(p) = \xi_T(p) = \xi(p)$ is merely a consequence of TFFH and temporal and spatial factorisation. On the other hand, deviations from scaling, whether of temporal or spatial type, indicate that some of the assumptions, TFFH and factorisation into universal and non-universal parts are violated. Therefore, deviations from spatial or temporal scaling are necessarily non-universal under the constraint of TFFH and temporal and spatial factorisation. The following argument demonstrates this statement.

Suppose we want to describe universal deviations from scaling (5) and (7) by the Ansatz

$$S_s^{(p)}(l) = c_s(p, U) l^{\xi_s(p,l)} \tag{17}$$

and

$$S_T^{(p)}(l) = c_T(p, U) l^{\xi_T(p,l)} \tag{18}$$

where the exponents ξ are assumed to be independent of U. This seems to be reasonable and for fixed U, delivers a description of experimentally observed structure functions, where $\xi_s(p) = \xi_s(p,l)$ and $\xi_T(p) = \xi_T(p,l)$ are not constant, but slowly varying with l. The above analysis showed that such a description contradicts the validity of TFFH. It is not possible for the functions $\xi_s(p,l)$ and $\xi_T(p,l)$ to be universal in the sense that they are independent of U. Universality is only possible for $\xi_s(p,l) = \xi_T(p,l) = \xi(p) =$ constant, i.e. a special case of scaling. The same argument also holds for $\overline{\epsilon}_s^{(p)}$ and $\overline{\epsilon}_T^{(p)}$.

For non-constant scaling exponents $\xi_s(p,U,l)$ and $\xi_T(p,U,l)$, we have to explicitely include some dependence on U. In this case it is important to know, whether it is then possible to consider cases where $\xi_s(p,U,l) = \xi_T(p,U,l)$. From an experimentally motivated point of view, this last relation insures that a temporal scaling analysis with (non-universal) functions $\xi_T(p,U,l)$ is equivalent to a spatial scaling analysis. The Ansatz

$$S_T^{(p)}(l) = c_T(p,U) l^{\xi(p,U,l)}, \tag{19}$$

$$S_s^{(p)}(l) = c_s(p,U) l^{\xi(p,U,l)} \tag{20}$$

together with (13) results in

$$\frac{c_T(p,U)}{c_s(p,U)} = \frac{(Ul)^{\xi(p,U,Ul)}}{l^{\xi(p,U,l)}} \tag{21}$$

where the right hand side is independent of l. A straightforward calculation shows that this is only possible for

$$\xi(p,U,l) = \frac{h(p,U)}{\ln l} \tag{22}$$

for some function h independent of l. In this case $S_T^{(p)}(l)$ and $S_s^{(p)}(l)$ are independent of l, which is the trivial case. Thus we necessarily have

$$\xi_T(p,U,\cdot) \neq \xi_s(p,U,\cdot). \tag{23}$$

Similar arguments hold for τ_s and τ_T.

The temporal scaling analysis of time series reveals the functions $\xi_T(p,U,l)$ and $\tau_T(p,U,l)$ for some fixed experimental set-up, i.e. for some fixed U. The range of arguments l where these scaling functions vary slowly and are approximately constant $\tau_T(p,U,l) \approx \tau_T(p,U)$, $\xi_T(p,U,l) \approx \xi_T(p,U)$ defines the inertial range. The constant approximations are then identified with their spatial counterpart. The above analysis showed that this procedure is correct, if and only if $\tau_T(p,U,l)$ and $\xi(p,U,l)$ do not vary with l and U. For TFFH to be true, we have $\tau_T(p,U,\cdot) \neq \tau_s(p,U,\cdot)$ and $\xi_T(p,U,\cdot) \neq \xi_s(p,U,\cdot)$. If TFFH is not exact, the procedure of analyzing time series to obtain information about spatial properties must be questioned from the very beginning.

IV. CONCLUSIONS

We showed that the violation of the scaling laws (5), (6) and (7), (8), as seen by observing scaling exponents τ and ξ that are not constant, implies scaling exponents to be non-universal and to depend on the experimental situation and the mean flow U. Furthermore, in that case $\tau_s \neq \tau_T$ and $\xi_s \neq \xi_T$., i.e. a temporal scaling analysis of τ_T and ξ_T does in general not allow to construct spatial exponents τ_s and ξ_s.

Deviations from scaling include non-universal features. In this respect it is worthwhile to comment on this non-universality with a view to possible non-universal mechanisms. An important point is that the most dominant deviations from scaling occur for the very small scales and the very large scales. In between, scaling holds at least approximately. The discussion of these deviations should focus on these small and large scales separately, since from a physical point of view they are different in nature. Large scale structures are dominated by special characteristics of the experiment like the geometry and boundary conditions of the experimental set-up, that introduces non-homogeneity. Thus universality and scaling is not expected to hold for these scales.

In the definition of the scaling relations (5), (6) and (7), (8) it is assumed that the flow is homogeneous, that structure functions do not depend on x and moments of the coarse grained energy dissipation do not depend on the location of the coarse graining domain. This is an important assumption about the flow. Together with the steadiness of the flow, it insures that the mean flow is constant and thus directly relates to TFFH. Focusing on spatial homogeneity only, it is clear that $\mathrm{E}\{\epsilon(x,t)\epsilon(x+\Delta x,t)\} \equiv \mathrm{D}(\Delta x)$ only depends on the distance Δx. This has important drawbacks for scaling relations. A simple example illustrates this. For $p=2$, equation (6) together with the assumed homogeneity yields

$$d_s(2,U)l^{2+\tau_s(2)} = \int_0^l (l-x)\,\mathrm{E}\left\{\epsilon(0,t)\epsilon(x,t)\right\}\mathrm{d}x. \tag{24}$$

If we differentiate this equation twice with respect to l we get

$$\mathrm{E}\left\{\epsilon(0,t)\epsilon(x,t)\right\} = d_s(2,U)\left(\tau_s(2)+2\right)\left(\tau_s(2)+1\right)x^{\tau_s(2)}. \tag{25}$$

Under the assumption of homogeneity the two-point expectations scale with the distance of the two points and the scaling exponent is equal to the second-order scaling exponent $\tau(2)$ of the coarse grained energy dissipation. For turbulence we have $\tau_s(2)<0$, but $\mathrm{E}\left\{\epsilon(x,t)^2\right\}<\infty$, which is clearly a contradiction. Thus the finiteness $\mathrm{E}\left\{\epsilon(x,t)^2\right\}<\infty$ necessarily leads to small-scale deviations from a scaling behaviour.

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[1] Barndorff-Nielsen, O.E., Eggers, H.C., Greiner, M. and Schmiegel, J. (2003): A Class of Spatio-Temporal and Causal Stochastic Processes, with Application to Multiscaling and Multifractality. Research Report 2003-33. MaPhySto, University of Aarhus.

[2] Barndorff-Nielsen, O.E. and Schmiegel, J. (2003): Lévy-based Tempo-Spatial Modelling; with Applications to Turbulence. Uspekhi Mat. Nauk **159** 63.

[3] Barndorff-Nielsen, O.E., Blæsild, P. and Schmiegel, J. (2004): A parsimonious and universal description of turbulent velocity increments. *Eur. Phys. J. B* **41**, 345-363.

[4] Frisch, U. (1995): *Turbulence. The legacy of A.N. Kolmogorov.* Cambridge University Press.

[5] Harris, S.R., Lempert, W.R., Hersch, L., Burchham, C.L., Saville, D.A., Miles, R.B., Gee, K. and Haugland, R.P. (1996): Quantitative measurements of internal circulation in droplets using flow tagging velocimetry. *AIAA J.* **34**, 449-454.

[6] Monin, A.S. and Yaglom, A.M. (1971): *Statistical Fluid Mechanics*, Vols. 1 and 2. Cambridge, MS: MIT Press.

[7] Navier, C.L.M.H. (1823): Mémoire sur les lois du mouvement des fluides. *Mém. Acad. Roy. Sci.* **6**, 389-440.

[8] Noullez, A., Wallace, G., Lempert, W., Miles, R.B. and Frisch, U. (1997): Transverse velocity increments in turbulent flow using the RELIEF technique. *J. Fluid Mech.* **339**, 287-307.

[9] Kolmogorov, A.N. (1941a): The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Dokl. Akad. Nauk. SSSR* **30**, 9-13.

[10] Kolmogorov, A.N. (1941b): Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk. SSSR* **32**, 16-18.

[11] Lempert, W.R., Magee, K., Ronney, P., Gee, K.R. and Haugland, R.P. (1995): Flow tagging velocimetry in incompressible flow using photo-activated non-intrusive tracking of molecular motion (PHANTOMM). *Exps. Fluids* **18**, 249-257.

[12] Pinton, J.F. and Labbé, R. (1994): Correction to the Taylor hypothesis in swirling flows. *J. Phys. II France* **4**, 1461-1468.

[13] Schmiegel, J., Dziekan, T., Cleve, J., Jouault, B. and Greiner, M. (2002): Scaling functions in a random multiplicative energy-cascade model of turbulence. Unpublished.

[14] Schmiegel, J., Cleve, J., Eggers, H.C., Pearson, B.R. and Greiner, M. (2004): Stochastic energy-cascade model for (1+1)-dimensional fully developed turbulence. *Phys.Lett. A* **320**, 247-253.

[15] Sreenivasan, K.R. (1991): Fractals and multifractals in fluid turbulence. *Ann. Rev. Fluid Mech.* **23**, 539-600.

[16] Sreenivasan, K.R. and Antonia, R.A. (1997): The phenomenology of small-scale turbulence. *Ann. Rev. Fluid Mech.* **29**, 435-472.

[17] Taylor, G.I. (1938): The spectrum of turbulence. *Proc. R. Soc. Lond. A* **164**, 476-490. **23**, 539-600.

[18] Wereley, S.T., Lueptow, R.M. (1998): Spatio-temporal character of non-wavy and wavy Taylor-Couette flow. *J. Fluid Mech.* **364**, 59-80.