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Brownian Semistationary Processes and Volatility/Intermittency

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Abstract. A new class of stochastic processes, termed Brownian semistationary processes ($\mathcal{BSS}$), is introduced and discussed. This class has similarities to that of Brownian semimartingales ($\mathcal{BSM}$), but is mainly directed towards the study of stationary processes, and $\mathcal{BSS}$ processes are not in general of the semimartingale type. We focus on semimartingale - nonsemimartingale issues and on inference problems concerning the underlying volatility/intermittency process, in the nonsemimartingale case and based on normalised realised quadratic variation. The concept of $\mathcal{BSS}$ processes has arisen out of an ongoing study of turbulent velocity fields and is the purely temporal version of the general tempo-spatial framework of ambit processes. The latter, which may have applications also to the finance of energy markets, is briefly considered at the end of the paper, again with reference to the question of inference on the volatility/intermittency.

Key words. Ambit Processes, Intermittency, Nonsemimartingales, Stationary Processes, Realised Quadratic Variation, Turbulence, Volatility

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0.1 Introduction

This paper discusses stochastic processes $Y = \{Y_t\}_{t\in\mathbb{R}}$ of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s \tag{0.1}$$

where μ is a constant, B is Brownian motion, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are càdlàg processes. When σ and a are stationary then so is Y. Accordingly we shall refer to processes of this type as *Brownian semistationary* ($\mathcal{BSS}$) processes. It is sometimes convenient to indicate the formula for Y as

$$Y = \mu + g * \sigma \bullet B + q * a \bullet Leb. \tag{0.2}$$

where Leb denotes Lebesgue measure.

We consider the $\mathcal{BSS}$ processes to be the natural analogue, for stationarity related processes, of the class $\mathcal{BSM}$ of Brownian semimartingales

$$Y_t = \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t a_s \mathrm{d}s. \tag{0.3}$$

In the present paper the processes σ and a will, unless otherwise stated, be taken to be stationary, and we then refer to σ as the *volatility* or *intermittency* process. The term intermittency comes from turbulence, and in that scientific field intermittency plays a key role, similar to that of (stochastic) volatility in finance.

In turbulence the basic notion of intermittency refers to the fact that the energy in a turbulent field is unevenly distributed in space and time. The present paper is part of a project with aim to construct a stochastic process model of the field of velocity vectors representing the fluid motion, conceiving of the intermittency as a positive random field with values $\sigma_t(x)$ at positions (x,t) in space-time. However, most extensive data sets on turbulent velocities only provide the time series of the main component (i.e. the component in the main direction of the fluid flow) of the velocity vector at a single location in space. In the present paper the focus is on this latter case, but in Sections 0.8 and 0.9 some discussion will be given on the further intriguing issues that arise when addressing tempo-spatial settings. For a detailed discussion of $\mathcal{BSS}$ and the more general concept of tempo-spatial ambit processes, in the context of turbulence modelling, we refer to Barndorff-Nielsen and Schmiegel (2004), Barndorff-Nielsen and Schmiegel (2007), Barndorff-Nielsen and Schmiegel (2008a), Barndorff-Nielsen and Schmiegel (2008b) and Barndorff-Nielsen and Schmiegel (2008c). There it is shown that such processes are able to reproduce main stylized facts of turbulent data.

In general, as we shall discuss in Section 0.3, models of the $\mathcal{BSS}$ form are not semimartingales. One consequence of this is that various useful techniques developed for semimartingales, such as the calculation of quadratic variation by Ito algebra and those of multipower variation, need extension or modification.

The recently established theory of multipower variation (Barndorff-Nielsen et al (2006a), Barndorff-Nielsen et al (2006b) and Jacod (2008a), cf. also Barndorff-Nielsen and Shephard (2003), Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen and Shephard (2006a), Barndorff-Nielsen and Shephard (2006b), Barndorff-Nielsen et al (2006c) and Jacod (2008b)) was developed as a basis for inference on σ under $\mathcal{BSM}$ models and, more generally Ito semimartingales, with particular focus on inference about the integrated squared volatility σ^{2+} given by

$$\sigma_t^{2+} = \int_0^t \sigma_s^2 \mathrm{d}s. \tag{0.4}$$

In the present paper the focus is similarly on inference for σ_t^{2+}. Specifically we shall discuss to what extent (a suitable normalised version of) realised quadratic variation of Y can be used to estimate σ_t^{2+}.

It is important to realise that, as regards inference on σ^{2+}, there may be substantial differences between cases where g is positive on all of $(0,\infty)$ and those where $g(t) = 0$ for $t > l$ for some $l \in (0,\infty)$. This will be discussed in detail later.

In semimartingale theory the quadratic variation $[Y]$ of Y is defined in terms of the Ito integral $Y \bullet Y$, as $[Y] = Y^2 - 2Y \bullet Y$. In that setting $[Y]$ equals the limit in probability as $\delta \to 0$ of the *realised quadratic variation* $[Y_\delta]$ of Y defined by

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(Y_{j\delta} - Y_{(j-1)\delta}\right)^2 \tag{0.5}$$

where $\lfloor t/\delta \rfloor$ is the largest integer smaller than or equal to t/δ.

However, the question of whether $[Y_\delta]$ has a limit in probability - and what that limit is - is of interest more broadly than for semimartingales, and in particular for $\mathcal{BSS}$ processes. For any process $Y = \{Y_t\}_{t\geq 0}$ we shall use $[Y.]$ to denote the limit, when it exists, i.e.

$$[Y.]_t = p\text{-}\lim_{\delta\to 0} [Y_\delta]_t\,.$$

Thus, in case $Y \in \mathcal{BSM}$ we have $[Y.] = [Y]$.[1]

We abbreviate realised quadratic variation to *RQV* and write *QV* for $[Y.]$.

The paper is organised as follows. Brownian semistationary processes are introduced in Section 0.2 and related non-semimartingale issues are considered in Section 0.3. Section 0.4 introduces a concept of q-orthogonality of stochastic processes and considers the computation of QV in some semimartingale difference cases. In Section 0.5 we turn to the increments of Brownian semistationary processes. Section 0.6 defines a normalised version $\overline{[Y_\delta]}$ of RQV, and Section 0.7 derives sufficient conditions for the convergence in probability of $\overline{[Y_\delta]}$ to σ^{2+}. Extensions to the tempo-spatial setting is discussed in Sectione 0.8 and 0.9. Some indications of ongoing further work and open problems are given in the concluding Section 0.10.

0.2 $\mathcal{BSS}$ processes

We have defined the concept of Brownian semistationary processes ($\mathcal{BSS}$) as processes $Y = \{Y_t\}_{t\in\mathbb{R}}$ of the form

$$Y_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^t q(t-s)a_s \mathrm{d}s \tag{0.6}$$

where, in the context of the present paper, the processes σ and a are taken to be stationary. The integrals in (0.6) are to be understood as limits in probability for $u \to -\infty$ of the integrals

$$\int_u^t g(t-s)\sigma_s \mathrm{d}B_s + \int_u^t q(t-s)a_s \mathrm{d}s$$

which are assumed to exist, the first defined for each fixed t as an Ito integral. This of course poses restrictions on which functions g and q are feasible, including square integrability of g.

The focus of the present paper is on inference about the integrated squared volatility σ^{2+} given by (0.4). In particular, we shall discuss to what extent realised quadratic variation of Y can be used to estimate σ_t^{2+}. Note that the relevant question here is whether a suitably normalised version of the realised quadratic variation, and not necessarily the realised quadratic variation itself, converges in probability and law.

As a modelling framework for continuous time stationary processes the specification (0.6) is quite general. In fact, the continuous time Wold-Karhunen decomposition says[2]

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[1] Of course, for semimartingales Y we have the more general result that

$$[Y]_t = p\text{-}\lim_{|\tau|\to 0} [Y_\tau]$$

where τ denotes a subdivision of $[0,t]$, $|\tau|$ is the maximal span in the subdivision, and Y_τ is the τ-discretisation of Y over the interval $[0,t]$.

[2] See Doob (1953) and Karhunen (1950)

that any second order stationary stochastic process, possibly complex valued, of mean 0 and continuous in quadratic mean can be represented as

$$Z_t = \int_{-\infty}^{t} \phi\left(t-s\right) \mathrm{d}\Xi_s + V_t$$

where

- the deterministic function ϕ is an, in general complex, deterministic square integrable function
- the process Ξ has orthogonal increments with $\mathrm{E}\left\{\left|\mathrm{d}\Xi_t\right|^2\right\} = \varpi \mathrm{d}t$ for some constant $\varpi > 0$
- the process V is nonregular (i.e. its future values can be predicted by linear operations on past values without error).

Under the further condition that $\cap_{t\in\mathbb{R}}\overline{\mathrm{sp}}\left\{Z_s : s \leq t\right\} = \{0\}$, the function ϕ is real and uniquely determined up to a real constant of proportionality; and the same is therefore true of Ξ (up to an additive constant).

0.3 $\mathcal{BSS}$ and semi - nonsemimartingale issues

If $Y \in \mathcal{BSS}$ then Y may or may not be of the semimartingale type. This Section discusses criteria for either of these cases.

0.3.1 Semimartingale cases

We begin by recalling a classical necesssary and sufficient condition, due to Knight (1992), for the process Y to be a semimartingale, valid in the special simple situation where $\sigma = 1$ and $a = 0$, i.e. where the process is of the form

$$Y_t = \int_{-\infty}^{t} g\left(t-s\right) \mathrm{d}B_s. \tag{0.7}$$

Knight's Theorem says that $\left(Y_t\right)_{t\geq 0}$ is a semimartingale in the $\left(\mathcal{F}_t^B\right)_{t\geq 0}$ filtration if and only if

$$g\left(t\right) = c + \int_0^t b\left(s\right) \mathrm{d}s \tag{0.8}$$

for some $c \in \mathbb{R}$ and a square integrable function b.

Example An example of some particular interest is where

$$g\left(t\right) = t^{\alpha} e^{-\lambda t} \quad \text{for} \quad t \in (0, \infty)$$

and some $\lambda > 0$. In order for the integral (0.7) to exist, α is required to be greater than $-\frac{1}{2}$, and for g to be of the form (0.8) we must have $\alpha = 0$ or $\alpha > \frac{1}{2}$. In other words, the nonsemimartingale cases are $\alpha \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$. $\square$

Generally, one may ask under what conditions moving average processes of the form

$$X_t = \int_{-\infty}^{\infty} \left(g\left(t-s\right) - h\left(-s\right)\right) \mathrm{d}B_s,$$

with g and h deterministic, are semimartingales. More specifically, when is $(X_t)_{t\geq 0}$ a $\left(\mathcal{F}_t^X\right)_{t\geq 0}$-seminartingale, where $\mathcal{F}_t^X$ is the σ-algebra generated by $\{X_s, s \leq t\}$. Constructive necessary and sufficient conditions for this are given in a recent paper by Basse, see Basse (2007a).

More generally still is the question of when a process X is a Gaussian semimartingale. Also for this case a necessary and sufficient criterion has been obtained by Basse, in Basse (2008), cf. also Basse (2007b) which discusses the spectral representation of Gaussian semimartingales.

At a further level of generalisation, Basse and Pedersen, in Basse and Pedersen (2008), consider processes X of the general form

$$X_t = \int_{-\infty}^{t} \left(\phi\left(t-s\right) - \psi\left(-s\right)\right) \mathrm{d}L_s$$

where L is a (two-sided) nondeterministic Lévy process with characteristic triplet $\left(\gamma, \sigma^2, \nu\right)$, ϕ and ψ are deterministic functions and the integral exists, in the sense of Rajput and Rosinski (1989). These authors establish various necessary conditions on $\left(\gamma, \sigma^2, \nu\right)$ and ϕ, ψ in order for $(X_t)_{t\geq 0}$ to be an $\left(\mathcal{F}_t^L\right)_{t\geq 0}$-semimartingale.

Now, turning to the general $\mathcal{BSS}$ case, we first argue formally, as if the differential of Y exists. From (0.6),

$$\mathrm{d}Y_t = g\left(0+\right)\sigma_t \mathrm{d}B_t + \{\dot{g} * \sigma \bullet B_t + q\left(0+\right) a_t + \dot{q} * a \bullet Leb_t\}\mathrm{d}t$$

suggesting that Y_t can be reexpressed as

$$Y_t = Y_0 + g\left(0+\right)\int_0^t \sigma_s \mathrm{d}B_s + \int_0^t A_s \mathrm{d}s$$

with

$$A = \dot{g} * \sigma \bullet B + q\left(0+\right) a + \dot{q} * a \bullet Leb.$$

This will indeed be the case provided the following conditions are satisfied (recall that we have assumed that σ and a are stationary):

(i) $g\left(0+\right)$ and $q\left(0+\right)$ exist and are finite.

(ii) g is absolutely continuous with square integrable derivative $\dot{g}$

(iii) The process $\dot{g}(-\cdot)\sigma_\cdot$ is square integrable

(iv) The process $\dot{q}(-\cdot)a_\cdot$ is integrable.

In view of the results by Knight and Basse, mentioned above, these conditions must be close to necessary as well.

We shall here not further discuss affirmative conditions for Y to be of the semimartingale type. Instead we turn to cases where Y can be written as a linear combination of semimartingales which are orthogonal, in a sense that will be specified, and have different filtrations.

0.4 RQV and linear combinations of semimartingales

While the focus will be on cases where a given $\mathcal{BSS}$ process Y can be rewritten as $Y^+ - Y^-$, where both Y^+ and Y^- are semimartingales, we begin by considering the broader issue of existence and calculation of $[Y_\cdot]$ when Y is a linear combination of q-orthogonal processes, q-orthogonality being defined below.

0.4.1 General considerations

Suppose that a process $Y = \{Y_t\}$ is representable in law as a linear combination $Y = Y' + Y''$ of some processes Y' and Y'' of interest, of semimartingale type or not. Then, defining $[Y'_\delta, Y''_\delta]$ and $[Y'_\cdot, Y''_\cdot]$ by

$$[Y'_\delta, Y''_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(Y'_{j\delta} - Y'_{(j-1)\delta} \right) \left(Y''_{j\delta} - Y''_{(j-1)\delta} \right)$$

and

$$[Y'_\cdot, Y''_\cdot] = p\text{-}\lim_{\delta \to 0} [Y'_\delta, Y''_\delta]_t$$

we have

$$[Y_\delta] = [Y'_\delta] + [Y''_\delta] + 2\,[Y'_\delta, Y''_\delta]$$

and hence, provided the limit exists (in probability),

$$[Y_\cdot] = [Y'_\cdot] + [Y''_\cdot] + 2\,[Y'_\cdot, Y''_\cdot]\,.$$

We will write this symbolically as

$$d\,[Y_\cdot] = \mathrm{d}\,[Y'_\cdot] + \mathrm{d}\,[Y''_\cdot] + 2\mathrm{d}\,[Y'_\cdot, Y''_\cdot]\,.$$

In case $[Y'_\cdot, Y''_\cdot] = 0$ we say that Y' and Y'' are *q-orthogonal* and express this by writing

$$\mathrm{d}Y'\mathrm{d}Y'' = 0.$$

Then

$$[Y_\cdot] = [Y'_\cdot] + [Y''_\cdot]\,.$$

In particular, if Y' and Y'' are both semimartingales, in general with different own filtrations, and q-orthogonal then

$$[Y_\cdot] = [Y] = [Y'] + [Y'']$$

and $\mathrm{d}\,[Y_\cdot]$ may be calculated as

$$\mathrm{d}\,[Y_\cdot]_t = \left(\mathrm{d}Y_t'\right)^2 + \left(\mathrm{d}Y_t''\right)^2.$$

In this case we may define $\mathrm{d}Y$ as $\mathrm{d}Y' + \mathrm{d}Y''$ and then, as in the usual semimartingale world, we have

$$[Y_\cdot]_t = \int_0^t \left(\mathrm{d}Y_s\right)^2 \mathrm{d}s.$$

An elementary instance of this $Y_t = Y_t' + Y_t''$ with $Y_t' = B_t$ and $Y_t'' = -B_{t-1}$ and where $B = \{B_t\}_{\mathbb{R}}$ is Brownian motion on the real line.

These considerations are extendable to settings where Y is a linear combination $Y = \int Y_t^{(c)} M\left(\mathrm{d}c\right)$ of mutually q-orthogonal processes $Y^{(c)}$ and where M is a deterministic, possibly signed, measure. We shall not here discuss specific general conditions for this; however an example is given in the next Subsection.

0.4.2 Some $\mathcal{BSS}$ cases

Let $\mathfrak{G}$ be the class of functions of the form (0.8). If $g \in \mathfrak{G}$ then for any $u > 0$ the function $h\left(\cdot\right) = g\left(\cdot + u\right)$ also belongs to $\mathfrak{G}$. This has the important consequence that if g is of the form $g^{\#}1_A$ with $A = (0, l)$ for some $l > 0$ and $g^{\#} \in \mathfrak{G}$ then Y itself is not a semimartingale but it is the difference between two semimartingales, specifically

$$Y_t = Y_t^+ - Y_t^-$$

where

$$Y_t^+ = \mu + \int_{-\infty}^t g\left(t-s\right)\sigma_s \mathrm{d}B_s + q * a \bullet Leb$$

and

$$Y_t^- = \int_{-\infty}^t g\left(t-s+l\right)\sigma_{s-l}\mathrm{d}B_{s-l}.$$

Both Y^+ and Y^- are semimartingales but with different filtrations, namely $\left\{\mathcal{F}_t^B\right\}_{t\in\mathbb{R}}$ and $\left\{\mathcal{F}_{t-l}^B\right\}_{t\in\mathbb{R}}$. Moreover, Y^+ and Y^- are q-orthogonal, and hence

$$\mathrm{d}\,[Y_\cdot]_t = \left(\mathrm{d}Y_t^+\right)^2 + \left(\mathrm{d}Y_t^-\right)^2.$$

More generally, suppose that g has the form

$$g\left(t\right) = \int_0^\infty g_0\left(t-c\right)\mathrm{d}M\left(c\right)$$

for a $g_0 \in \mathfrak{G}$ and where M is a function of bounded variation on $\mathbb{R}_+$. In this case we have

$$\begin{aligned} Y_t &= \int_{-\infty}^{t} g(t-s)\,\sigma_s \mathrm{d}B_s \\ &= \int_{-\infty}^{t}\int_{0}^{\infty} g_0(t-s-c)\,\mathrm{d}M(c)\,\sigma_s \mathrm{d}B_s \\ &= \int_{0}^{\infty}\int_{-\infty}^{t} g_0(t-s-c)\,\sigma_s \mathrm{d}B_s \mathrm{d}M(c) \\ &= \int_{0}^{\infty}\int_{-\infty}^{t-c} g_0(t-c-s)\,\sigma_s \mathrm{d}B_s \mathrm{d}M(c) \\ &= \int_{0}^{\infty} Y_t^{(c)} \mathrm{d}M(c) \end{aligned}$$

where

$$Y_t^{(c)} = \int_{-\infty}^{t} g_0(t-s)\,\sigma_{s-c} \mathrm{d}B_{s-c}$$

showing that Y is a linear combination of q-orthogonal semimartingales with different filtrations (namely, conditional on σ the filtration of $Y^{(c)}$ is $\left\{\mathcal{F}_{t-c}^{B}\right\}_{t\in\mathbb{R}}$.)

0.5 Increment processes

Again, suppose that $g = g^{\#}1_{[0,l]}$ for some $l > 0$ and $g^{\#} \in \mathfrak{G}$. For any given t we define the increment process $\left\{X_{u|t}\right\}_{u\geq 0}$ by

$$\begin{aligned} X_{t+u|t} &= Y_{t+u} - Y_t \\ &= \int_{t}^{t+u} g(t+u-s)\,\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} \{g(t+u-s) - g(t-s)\}\,\sigma_s \mathrm{d}B_s \\ &\quad + \int_{t}^{t+u} q(t+u-s)\,a_s \mathrm{d}s + \int_{-\infty}^{t} \{q(t+u-s) - q(t-s)\}\,a_s \mathrm{d}s. \end{aligned}$$

It will be convenient to rewrite $X_{t|t-u}$ as

$$X_{t|t-u} = \int_{-\infty}^{0} \phi_u(-v)\,\sigma_{v+t} \mathrm{d}B_{v+t} + \int_{-\infty}^{0} \chi_u(-v)\,a_{v+t} \mathrm{d}v \qquad (0.9)$$

where ϕ_u and χ_u are defined by

$$\phi_u(v) = \begin{cases} g(v) & \text{for} \quad 0 \leq v < u \\ g(v) - g(v-u) & \text{for} \quad u \leq v < \infty \end{cases}$$

and

$$\chi_u(v) = \begin{cases} q(v) & \text{for} \quad 0 \leq v < u \\ q(v) - q(v-u) & \text{for} \quad u \leq v < \infty \end{cases}.$$

From now on we assume that (σ, a) is independent of B and that a is adapted to the filtration $\mathcal{F}^\sigma$. This, together with (0.9), implies in particular that the conditional variance of $Y_t - Y_{t-u}$ given the process σ takes the form

$$\mathrm{E}\left\{(Y_t - Y_{t-u})^2 |\sigma\right\} = \int_0^\infty \psi_u(v)\,\sigma_{t-v}^2 \mathrm{d}v + \left(\int_0^\infty \chi_u(v)\,a_{t-v}\mathrm{d}v\right)^2$$

where

$$\psi_u(v) = \begin{cases} g^2(v) & \text{for} \quad 0 \le v < u \\ \{g(v-u) - g(v)\}^2 & \text{for} \quad u \le v < \infty \end{cases}.$$

Remark 1 Note that $\phi_u(v) = \psi_u(v) = \chi_u(v) = 0$ for $v \ge l + u$ while for $l \le v < l + u$ we have $\psi_u(v) = g(v-u)^2$ and $\chi_u(v) = q(v-u)$. □

Let

$$c(u) = \int_0^\infty \psi_u(v)\,\mathrm{d}v. \tag{0.10}$$

Remark 2 Trivially,

$$c(\delta) \ge \int_0^\delta \psi_\delta(v)\,\mathrm{d}v = \int_0^\delta g^2(v)\,\mathrm{d}v$$

implying that if $g(0+) > 0$ then $c(\delta)$ cannot tend to 0 faster than δ. □

Remark 3 We have

$$c(u) = 2\,\|g\|^2\,\bar{r}(u)$$

where $\bar{r} = 1 - r$ with r being the autocorrelation function of Y. Furthermore,

$$\begin{aligned} \mathrm{E}\left\{(Y_t - Y_{t-u})^2\right\} &= \mathrm{E}\left\{\sigma_0^2\right\} c(u) \\ &\quad + \mathrm{E}\left\{a_0^2\right\} \int_0^\infty \int_0^\infty \chi_u(v)\,\chi_u(w)\,\varrho(|v-w|)\,\mathrm{d}v\mathrm{d}w \end{aligned}$$

where ϱ is the autocorrelation function of a. □

0.6 Normalised RQV

We now define the *normalised RQV* as

$$\overline{[Y_\delta]} = \frac{\delta}{c(\delta)}\,[Y_\delta]\,.$$

The question we wish to address here is whether and under what conditions $\overline{[Y_\delta]}$ converges in probability to σ^{2+}. Concerning the related question of a central limit theorem for $\overline{[Y_\delta]}$, see Section 0.10.

In the present paper we shall largely restrict the discussion to quite regular forms of the weight function g, assuming in particular that g is positive on a finite interval $(0, l)$ only. Specifically, we now assume that the function g is positive, continuously differentiable, convex and decreasing on an interval $(0, l)$ where $0 < l < \infty$ and that $g(t) = 0$ outside that interval. Also, we require that σ and a are stationary and càdlàg and, as before, that a is adapted to the natural filtration of σ. Without loss of generality we take t/δ to be an integer n so that $t = n\delta$. Below C denotes a constant that is independent of n but whose value may change with the context.

0.7 Consistency

To discuss the question of when $\overline{[Y_\delta]} \xrightarrow{p} \sigma^{2+}$ we first note that, by (0.9),

$$\begin{aligned}[Y_\delta]_t &= \sum_{k=1}^{n}\left(\int_{-\infty}^{0}\phi_\delta(-v)\,\sigma_{v+k\delta}\mathrm{d}B_{v+k\delta}\right)^2 \\ &\quad +2\sum_{k=1}^{n}\int_{-\infty}^{0}\phi_\delta(-v)\,\sigma_{v+k\delta}\mathrm{d}B_{v+k\delta}\int_{-\infty}^{0}\chi_\delta(-v)\,a_{v+k\delta}\mathrm{d}v \\ &\quad +\sum_{k=1}^{n}\left(\int_{-\infty}^{0}\chi_\delta(-v)\,a_{v+k\delta}\mathrm{d}v\right)^2.\end{aligned}$$

It follows that

$$\mathrm{E}\left\{\overline{[Y_\delta]_t}|\sigma\right\} = \int_0^\infty\left\{\delta\sum_{k=1}^{n}\sigma^2_{k\delta-v}\right\}\pi_\delta(\mathrm{d}v) + c(\delta)^{-1}D_\delta(a) \tag{0.11}$$

where

$$\pi_\delta(\mathrm{d}v) = \frac{\psi_\delta(v)}{c(\delta)}\mathrm{d}v$$

and

$$D_\delta(a) = \int_0^\infty\int_0^\infty\chi_\delta(v)\,\chi_\delta(w)\left\{\delta\sum_{k=1}^{n}a_{k\delta-v}a_{k\delta-w}\right\}\mathrm{d}v\mathrm{d}w.$$

Thus π_δ is an absolutely continuous probability measure on $(0, l+\delta)$. Furthermore,

$$|D_\delta(a)| \le C\left(\int_0^\infty|\chi_\delta(v)|\,\mathrm{d}v\right)^2$$

where the constant C depends on a,l and t.

This leads us to introduce

Condition A $c(\delta)^{-1}\left(\int_0^\infty|\chi_\delta(v)|\,\mathrm{d}v\right)^2 \to 0.$ □

Remark 4 Note that in this connection if q is a positive decreasing function then

$$\int_0^\infty |\chi_\delta(v)|\,\mathrm{d}v = 2\int_0^\delta q(v)\,\mathrm{d}v. \tag{0.12}$$

□

Suppose that π_δ converges weakly, as $\delta \to 0$, to a probability measure π on $[0,l]$, i.e.

$$\pi_\delta \stackrel{w}{\to} \pi. \tag{0.13}$$

Then, if Condition A holds we obtain from (0.11) that

$$\mathrm{E}\left\{\overline{[Y_\delta]}_t|\sigma\right\} \to \int_0^\infty \left(\sigma^{2+}_{t-v} - \sigma^{2+}_{-v}\right)\pi(\mathrm{d}v). \tag{0.14}$$

In particular, if $\pi = \delta_0$, the delta measure at 0, then

$$\mathrm{E}\left\{\overline{[Y_\delta]}_t|\sigma\right\} \to \sigma^{2+}_t \tag{0.15}$$

where

$$\sigma^{2+}_t = \int_0^t \sigma^2_s \mathrm{d}s.$$

The following two Subsections derive sufficient conditions for $\pi_\delta \stackrel{w}{\to} \delta_0$ and for $\mathrm{Var}\left\{\overline{[Y_\delta]}_t|\sigma\right\} \to 0$, respectively. These two relations together with Condition A imply that

$$\overline{[Y_\delta]} \stackrel{p}{\to} \sigma^{2+}. \tag{0.16}$$

We will refer to the case where (0.16) is satisfied by saying that the model for Y is *volatility memoryless*.

0.7.1 pidelta to pi

Suppose that $l < \infty$ and let

$$\Psi_\delta(u) = \int_0^u \psi_\delta(v)\,\mathrm{d}v \quad \text{and} \quad \bar{\Psi}_\delta(u) = \int_{l+\delta-u}^{l+\delta} \psi_\delta(v)\,\mathrm{d}v,$$

so that $c(\delta)^{-1}\Psi_\delta$ is the distribution function, say Π_δ, of π_δ.

Next, for $k = 1, 2, ...$, let

$$\begin{aligned} c_k(\delta) &= \int_{(k-1)\delta}^{k\delta} \psi_\delta(u)\,\mathrm{d}u \\ &= \int_0^\delta \psi_\delta((k-1)\delta + u)\,\mathrm{d}u \\ &= \delta\int_0^1 \psi_\delta((k-1+u)\delta)\,\mathrm{d}u \end{aligned}$$

i.e.

$$c_k(\delta) = \delta \int_0^1 \{g((k-2+u)\delta) - g((k-1+u)\delta)\}^2 \, \mathrm{d}u. \tag{0.17}$$

We must now distinguish between the cases $t < l$ and $t \geq l$.

Suppose first that $t \geq l$. Let $k^* = \max\{k : k\delta \leq l\}$. Then, by (0.17), for $1 < k \leq k^*$

$$c_k(\delta) = \delta^3 \int_0^1 g'((k-2+u+\theta_k(u))\delta)^2 \, \mathrm{d}u$$

where the $\theta_k(u)$ satisfy $0 \leq \theta_k(u) \leq 1$. Since g is convex and decreasing this implies, provided $k_* \leq k \leq k^*$ where $k_* > 2$, that

$$c_k(\delta) \leq \delta^3 g'((k-2)\delta)^2 \leq \delta^3 g'((k_*-2)\delta)^2.$$

Therefore, for any $\varepsilon \in (2\delta, l)$ with $1 < \lfloor \varepsilon/\delta \rfloor < k^*$ we have

$$\begin{aligned} \Psi_\delta(k^*\delta) - \Psi_\delta(\varepsilon) &\leq \sum_{k=\lfloor \varepsilon/\delta \rfloor + 1}^{k^*} c_k(\delta) \\ &\leq \delta^3 (k^* - \lfloor \varepsilon/\delta \rfloor) g'(\lfloor \varepsilon - 2\delta \rfloor)^2 \\ &\leq (l - \varepsilon + \delta) g'(\lfloor \varepsilon - 2\delta \rfloor)^2 \delta^2 \end{aligned}$$

so that

$$\Pi_\delta(k^*\delta) - \Pi_\delta(\varepsilon) \leq (l - \varepsilon + \delta) g'(\lfloor \varepsilon - 2\delta \rfloor)^2 \delta^2 c(\delta)^{-1}.$$

Consequently, as $\delta \to 0$,

$$\Pi_\delta(k^*\delta) - \Pi_\delta(\varepsilon) \to 0.$$

It follows that if π_δ converges to a probability measure π then π is necessarily a linear combination of the delta measures at 0 and l.

Furthermore,

$$\Psi_\delta(l+\delta) - \Psi_\delta(k^*\delta) = c_{k^*+1}(\delta) + c_{k^*+2}(\delta)$$

where

$$\begin{aligned} c_{k^*+1}(\delta) &= \int_{k^*\delta}^{(k^*+1)\delta} \psi_\delta(v) \, \mathrm{d}v \\ &= \int_{k^*\delta}^{l} \{g(v-\delta) - g(v)\}^2 \, \mathrm{d}v \\ &\quad + \int_{l}^{(k^*+1)\delta} g^2(v-\delta) \, \mathrm{d}v \end{aligned}$$

and

$$c_{k^*+2}(\delta) = \int_{(k^*+1)\delta}^{l+\delta} g^2(v-\delta) \, \mathrm{d}v.$$

So, combining, for $\pi_\delta \to \delta_0$ to hold we must require that

$$c(\delta)^{-1} \int_{k^*\delta}^{l} \{g(v-\delta) - g(v)\}^2 \,\mathrm{d}v \to 0$$

and

$$c(\delta)^{-1} \int_{l}^{l+\delta} g^2(v-\delta)\,\mathrm{d}v \to 0.$$

But the first relation follows from the smoothness of g, so to guarantee $\pi_\delta \xrightarrow{w} \delta_0$, when $t \geq l$, we therefore only need to add

Condition B $c(\delta)^{-1} \int_l^{l+\delta} g^2(v-\delta)\,\mathrm{d}v \to 0$ as $\delta \downarrow 0$ $\square$

Remark 5 Condition B is equivalently to having

$$\frac{\int_{l-\delta}^{l} g^2(v)\mathrm{d}v}{\int_0^\delta g^2(v)\mathrm{d}v} \to 0,$$

as follows from the above discussion. In particular, it suffices to have $g(v) \to 0$ as $v \uparrow l$. $\square$

Remark 6 In case $c(\delta)^{-1} \int_l^{l+\delta} g^2(v-\delta)\,\mathrm{d}v \to \lambda \in (0,1)$ we obtain $\pi_\delta \to (1-\lambda)\delta_0 + \lambda\delta_1$. $\square$

When $t < l$, for any $\varepsilon \in (2\delta, t)$ with $1 < \lfloor \varepsilon/\delta \rfloor < n$,

$$\begin{aligned}\Psi_\delta(t+\delta) - \Psi_\delta(\varepsilon) &\leq \sum_{k=\lfloor \varepsilon/\delta \rfloor+1}^{n} c_k(\delta) \\ &\leq (t-\varepsilon+\delta)\, g'(\lfloor \varepsilon - 2\delta \rfloor)^2 \delta^2\end{aligned}$$

which tends to 0 at the order of δ^2. To obtain $\pi_\delta \xrightarrow{w} \delta_0$ we therefore only need to add the assumption that

$$\Psi_\delta(l+\delta) - \Psi_\delta(t+\delta) = o(c(\delta)). \tag{0.18}$$

Now,

$$\Psi_\delta(l+\delta) - \Psi_\delta(t+\delta) = \sum_{k=n+1}^{\infty} c_k(\delta).$$

Thus, letting

$$\bar{c}_k(\delta) = \frac{c_k(\delta)}{c(\delta)} \tag{0.19}$$

we have that (0.18) is implied by

Condition C $\sum_{k=n+1}^{\infty} \bar{c}_k(\delta) \to 0$ as $\delta \downarrow 0$. $\square$

0.7.2 Conditional Var to 0

We now establish conditions under which the conditional variance of the normalised realised quadratic variation tends to 0 as $\delta \to 0$, i.e.

$$\mathrm{Var}\{\overline{[Y_\delta]_t}|\sigma\} \to 0. \tag{0.20}$$

Suppose first that $a = 0$.

Let $\Delta_j^n Y = Y_{j\delta} - Y_{(j-1)\delta}$. Then

$$\mathrm{Var}\{\overline{[Y_\delta]_t}|\sigma\} = \frac{\delta^2}{c(\delta)^2}\left\{\sum_{j=1}^{n} \mathrm{Var}\{\left(\Delta_j^n Y\right)^2|\sigma\} + 2\sum_{j=1}^{n}\sum_{k=j+1}^{n} \mathrm{Cov}\{\left(\Delta_j^n Y\right)^2, \left(\Delta_k^n Y\right)^2|\sigma\}\right\}$$

where, for $j < k$,

$$\begin{aligned} \mathrm{Cov}\{\Delta_j^n Y \Delta_k^n Y|\sigma\} &= \mathrm{E}\left\{\left(Y_{j\delta} - Y_{(j-1)\delta}\right)\left(Y_{k\delta} - Y_{(k-1)\delta}\right)|\sigma\right\} \\ &= \int_0^\infty \phi_\delta\left(\left(k-j\right)\delta + u\right)\phi_\delta\left(u\right)\sigma_{j\delta-u}^2 \mathrm{d}u. \end{aligned}$$

Let $K(\sigma) = \sup_{-l\leq s\leq t}\sigma_s^2$. As σ is assumed càdlàg, $K(\sigma) < \infty$ a.s.. Hence, by the Cauchy-Schwarz inequality,

$$\left|\mathrm{Cov}\{Y_{j\delta} - Y_{(j-1)\delta}, Y_{k\delta} - Y_{(k-1)\delta}|\sigma\}\right| \leq K(\sigma)\left(\int_0^\infty \psi_\delta(u)\,\mathrm{d}u\right)^{1/2}\left(\int_{(k-j)\delta}^\infty \psi_\delta(u)\,\mathrm{d}u\right)^{1/2}.$$

Now, recall that for any pair X and Y of normal, mean zero random variables we have

$$\mathrm{Cov}\{X^2, Y^2\} = 2\mathrm{Cov}\{X, Y\}^2. \tag{0.21}$$

Therefore

$$\begin{aligned} \mathrm{Var}\{\overline{[Y_\delta]_t}|\sigma\} &\leq 2K(\sigma)^2\frac{\delta^2}{c(\delta)^2}\left(l\delta^{-1}c(\delta)^2 + 2c(\delta)\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\int_{(i-j)\delta}^\infty \psi_\delta(u)\,\mathrm{d}u\right) \\ &= 2K(\sigma)^2\,\delta\left(l + 2\frac{\delta}{c(\delta)}\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\int_{(i-j)\delta}^\infty \psi_\delta(u)\,\mathrm{d}u\right) \end{aligned}$$

Here

$$\begin{aligned}\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\int_{(i-j)\delta}^{\infty}\psi_\delta(u)\,\mathrm{d}u &= \sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\int_{i\delta}^{\infty}\psi_\delta(u)\,\mathrm{d}u \\ &= \sum_{\nu=1}^{n-1}\sum_{i=1}^{\nu}\sum_{k=i}^{\infty}c_{k+1}(\delta) \\ &= \sum_{\nu=1}^{n-1}\sum_{k=1}^{\infty}c_{k+1}(\delta)\sum_{i=1}^{\nu}1_{\le k}(i) \\ &= \sum_{\nu=1}^{n-1}\left(\sum_{k=1}^{\nu}kc_{k+1}(\delta)+\nu\sum_{k=\nu}^{\infty}c_{k+2}(\delta)\right) \\ &= \sum_{k=1}^{n-1}(n-k)\,kc_{k+1}(\delta)+\sum_{k=1}^{\infty}c_{k+2}(\delta)\sum_{\nu=1}^{k\wedge(n-1)}\nu \\ &= \sum_{k=1}^{n-1}\left\{(n-k)\,kc_{k+1}(\delta)+\frac{1}{2}(k+1)\,kc_{k+2}\right\} \\ &\quad +\frac{(n-1)n}{2}\sum_{k=n}^{\infty}c_{k+2}(\delta)\end{aligned}$$

With the notation (0.19) we thus have

$$\begin{aligned}\mathrm{Var}\{\overline{[Y_\delta]}_t|\sigma\} &\le 2K(\sigma)^2\delta\left(l+2\delta\sum_{k=1}^{n-1}(n-k)\,k\bar{c}_{k+1}(\delta)+2\delta\sum_{k=1}^{\infty}\bar{c}_{k+2}(\delta)\sum_{\nu=1}^{k\wedge(n-1)}\nu\right) \\ &= 2K(\sigma)^2 l\delta \\ &\quad +2K(\sigma)^2\delta^2\sum_{k=1}^{n-1}\left\{(n-k)\,k\bar{c}_{k+1}(\delta)+\frac{1}{2}(k+1)\,k\bar{c}_{k+2}\right\} \\ &\quad +2\delta K(\sigma)^2\,\frac{(n-1)\,n}{2}\sum_{k=n}^{\infty}\bar{c}_{k+2}(\delta)\,.\end{aligned}$$

Here

$$\delta^2\sum_{k=1}^{n-1}\left\{(n-k)\,k\bar{c}_{k+1}(\delta)+\frac{1}{2}(k+1)\,k\bar{c}_{k+2}\right\}\le C\delta\sum_{k=1}^{n}k\bar{c}_k(\delta)$$

and

$$\delta^2\frac{(n-1)\,n}{2}\sum_{k=n}^{\infty}\bar{c}_{k+2}(\delta)\le C\sum_{k=n+1}^{\infty}\bar{c}_k(\delta)\,.$$

Consequently, when $a = 0$, for (0.20) to be valid it suffices to have

$$\delta \sum_{k=1}^{n} k\bar{c}_k(\delta) \to 0 \quad \text{and} \quad \sum_{k=n+1}^{\infty} \bar{c}_k(\delta) \to 0. \tag{0.22}$$

Condition C will ensure the second limit result, and we now introduce

Condition D $\delta \sum_{k=1}^{n} k\bar{c}_k(\delta) \to 0$ as $\delta \downarrow 0$. $\square$

Provided $a = 0$, for (0.20) to be valid it suffices that Conditions C and D to hold.

Next we show that the convergence $\mathrm{Var}\{\overline{[Y_\delta]_t}|\sigma\} \to 0$ also holds if a is not 0 provided Condition A is fulfilled too. In case $a \neq 0$, $\mathrm{Var}\{\overline{[Y_\delta]_t}|\sigma\}$ is a sum of two terms, one as above for $a = 0$ while the other is

$$4\frac{\delta^2}{c(\delta)^2} \sum_{k=1}^{n} \int_0^\infty \psi_\delta(v)\,\sigma^2_{k\delta-v}\mathrm{d}v \left(\int_0^\infty \chi_\delta(v)\,a_{k\delta-v}\mathrm{d}v \right)^2 \tag{0.23}$$

which is bounded above by $4HK$ where

$$H = \limsup_{k,\delta} \frac{\delta}{c(\delta)} \int_0^\infty \psi_\delta(v)\,\sigma^2_{k\delta-v}\mathrm{d}v$$

and

$$K = \frac{\delta}{c(\delta)} \sum_{k=1}^{n} \left(\int_0^\infty \chi_\delta(v)\,a_{k\delta-v}\mathrm{d}v \right)^2.$$

Here

$$\int_0^\infty \psi_\delta(v)\,\sigma^2_{k\delta-v}\mathrm{d}v \le Cc(\delta)$$

where the constant C depends on t and σ. Hence $H \to 0$. Furthermore,

$$\sum_{k=1}^{n} \left(\int_0^\infty \chi_\delta(v)\,a_{k\delta-v}\mathrm{d}v \right)^2 \le C \sum_{k=1}^{n} \left(\int_0^\infty |\chi_\delta(v)|\,\mathrm{d}v \right)^2 = C\delta^{-1} \left(\int_0^\infty |\chi_\delta(v)|\,\mathrm{d}v \right)^2$$

where C, again, depends on t and a. Hence Condition A implies $K \to 0$.

0.7.3 Summing up

Suppose first that $t < l < \infty$, which is the most interesting case from the viewpoint of turbulence modelling. If

$$c(\delta)^{-1} \left(\int_0^\infty |\chi_\delta(v)|\,\mathrm{d}v \right)^2 \to 0 \tag{0.24}$$

$$\sum_{k=n+1}^{\infty} \bar{c}_k(\delta) \to 0 \tag{0.25}$$

and

$$\delta \sum_{k=1}^{n} k\bar{c}_k(\delta) \to 0 \tag{0.26}$$

then

$$\pi_\delta \to \delta_0, \mathrm{Var}\{\overline{[Y_\delta]}|\sigma\} \to 0 \text{ and } \overline{[Y_\delta]} \xrightarrow{p} \sigma^{2+}. \tag{0.27}$$

If $l \leq t$ then the additional assumption that

$$\frac{\int_{l-\delta}^{l} g^2(v)\mathrm{d}v}{\int_0^\delta g^2(v)\mathrm{d}v} \to 0 \tag{0.28}$$

is required. The latter is, in particular, fulfilled if $g(v) \to 0$ for $v \uparrow l$. In case (0.28) is violated but (0.24), (0.25) and (0.26) hold and $\pi_\delta \xrightarrow{w} \pi$ for some π, necessarily of the form $\pi = \lambda\delta_0 + (1-\lambda)\delta_l$ for some $\lambda \in (0,1)$, then

$$\overline{[Y_\delta]}_t \xrightarrow{p} \lambda\sigma_t^{2+} + (1-\lambda)\left(\sigma_{t-l}^{2+} - \sigma_{-l}^{2+}\right). \tag{0.29}$$

0.7.4 Examples

Recall Conditions A-D:

$$c(\delta)^{-1}\left(\int_0^\infty |\chi_\delta(v)|\,\mathrm{d}v\right)^2 \to 0 \tag{0.30}$$

$$c(\delta)^{-1}\int_l^{l+\delta} g^2(v-\delta)\,\mathrm{d}v \to 0$$

$$\sum_{k=n+1}^{\infty} \bar{c}_k(\delta) \to 0 \tag{0.31}$$

$$\delta \sum_{k=1}^{n} k\bar{c}_k(\delta) \to 0. \tag{0.32}$$

In this Section we suppose that $q = g$. Then Condition A has the form

$$c(\delta)^{-1} c_1(\delta)^2 \to 0. \tag{0.33}$$

Example Suppose that $t = l$ and $g(v) = e^{-\lambda v}1_{(0,l)}(v)$ (a non-semimartingale case). Then

$$\psi_\delta(v) = e^{-2\lambda v}\begin{cases} 1 & \text{for} \quad 0 \leq v < \delta \\ \left(e^{\lambda\delta} - 1\right)^2 & \text{for} \quad \delta \leq v < l \\ e^{2\lambda\delta} & \text{for} \quad l \leq v < l+\delta \\ 0 & \text{for} \quad l+\delta \leq v \end{cases}.$$

Here we find

$$c_1(\delta) = \frac{1}{2\lambda}\left(1 - e^{-2\lambda\delta}\right) \sim \delta$$

while for $k = 2, ..., n$

$$c_k(\delta) = \frac{1}{2\lambda}\left(e^{\lambda\delta} - 1\right)^3 e^{-2k\lambda} \sim \frac{\lambda^2}{2} e^{-2k\lambda}\delta^3.$$

Moreover we have

$$c_{n+1}(\delta) = \frac{1}{2\lambda}\left(1 - e^{-2\lambda\delta}\right) \sim e^{-2\lambda l}\delta,$$

whereas $c_k(\delta) = 0$ for $k > n+1$. Finally, $c(\delta) \sim \delta(1 + e^{-2\lambda l})$ and

$$\overline{c}_{n+1}(\delta) c(\delta)^{-1} \int_l^{l+\delta} g^2(v - \delta)\,\mathrm{d}v \to \left(1 + e^{2\lambda l}\right)^{-1}.$$

So, Conditions A, C and D are met. But Condition B is not and we have that $\pi_\delta \to \pi$, where

$$\pi = \frac{1}{1 + e^{-2\lambda l}}\delta_0 + \frac{1}{1 + e^{2\lambda l}}\delta_1,$$

and thus

$$\overline{[Y_\delta]} \xrightarrow{p} \sigma_t^{2+} - \left(1 + e^{2\lambda l}\right)^{-1} \sigma_{-t}^{2+}.$$

□

Example Let $g(v) = v^\alpha (1 - v)^\beta 1_{(0,1)}(v)$ with $-\frac{1}{2} < \alpha$ and $\beta \geq 1$. The first inequality ensures existence of the stochastic integral $g * \sigma \bullet B$, and if $\alpha < 0$ then we are in the nonsemimartingale situation. In showing that $\pi_\delta \to \delta_0$ and $\overline{[Y_\delta]} \xrightarrow{p} \sigma^{2+}$ it suffices to consider the case where $-\frac{1}{2} < \alpha < 0$, $\beta = 1$ and $n\delta = t$. Let $\gamma = -\alpha$, and suppose $t < 1$.

We find

$$\begin{aligned} c_0(\delta) &= \int_0^\delta u^{-2\gamma}(1 - u)^2\,\mathrm{d}u \\ &= (1 - 2\gamma)^{-1}\delta^{1-2\gamma}(1 + O(\delta)) \end{aligned}$$

and, for $k = 1, 2, ..., n-1$,

$$\begin{aligned} c_k(\delta) &= \delta \int_0^1 \left[((k+u)\delta)^{-\gamma} - ((k+u)\delta)^{1-\gamma} - ((k+u-1)\delta)^{-\gamma} + ((k+u-1)\delta)^{1-\gamma}\right]^2 \mathrm{d}u \\ &= \delta^{1-2\gamma} \int_0^1 \left[(k+u)^{-\gamma} - (k+u-1)^{-\gamma} - \delta\left\{(k+u)^{1-\gamma} - (k+u-1)^{1-\gamma}\right\}\right]^2 \mathrm{d}u \end{aligned}$$

while

$$\begin{aligned} c_n(\delta) &= \delta^{3-2\gamma} \int_0^1 \left[(n+u)^{1-\gamma} - (n+u-1)^{1-\gamma}\right]^2 \mathrm{d}u \\ &= \delta^{3-2\gamma} n^{2-2\gamma} \int_0^1 \left[\left(1 + \frac{u}{n}\right)^{1-\gamma} - \left(1 + \frac{u-1}{n}\right)^{1-\gamma}\right]^2 \mathrm{d}u \\ &= O\left(\delta^2\right) \end{aligned}$$

and $c_k(\delta) = 0$ when $k > n$. It follows, in particular, that

$$c_1(\delta) = O\left(\delta^{1-2\gamma}\right);$$

furthermore, since for $1 < k < n$ and $0 \le u \le 1$

$$\left|(k+u)^{-\gamma} - (k+u-1)^{-\gamma}\right| \le \gamma (k-1)^{-\gamma-1}$$

and

$$\left|(k+u)^{1-\gamma} - (k+u-1)^{1-\gamma}\right| \le (1-\gamma)(k-1)^{-\gamma}$$

we have (when $\delta < 1$)

$$\begin{aligned} c_k(\delta) &\le \delta^{1-2\gamma}(k-1)^{-2\gamma-2}\left[\gamma + (1-\gamma)\delta(k-1)\right]^2 \\ &\le \delta^{1-2\gamma}k^{-2\gamma-2}\left(1-\frac{1}{k}\right)^{-2\gamma-2} \\ &\le C\delta^{1-2\gamma}k^{-2\gamma-2}. \end{aligned}$$

Consequently,

$$c(\delta) = O\left(\delta^{1-2\gamma}\right)$$

while for $1 < k < n$

$$k\bar{c}_k(\delta) \le Ck^{-2\gamma-1}$$

so that

$$\sum_{1}^{n-1} k\bar{c}(\delta) \le C.$$

We conlude that the Conditions A-D are satisfied and hence that $\overline{[Y_\delta]}_t \xrightarrow{p} \sigma_t^{2+}$. $\square$

0.8 Tempo-spatial setting

Above only the case of time-wise behaviour at a single point in space was considered. In the real turbulence setting, space and the velocity vector are three dimensional. The general modelling framework specifies the velocity and intermittency fields as

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_t(x)} g(t-s, |\xi - x|)\,\sigma_s(\xi)\,W(\mathrm{d}\xi, \mathrm{d}s) \\ &\quad + \int_{C_t(x)} q(t-s, |\xi - x|)\,a_s(\xi)\,\mathrm{d}\xi\mathrm{d}s \end{aligned}$$

and

$$\sigma_t^2(x) = \int_{D_t(x)} h(t-s, |\xi - x|)\,L(\mathrm{d}\xi, \mathrm{d}s)$$

Here Y_t is a vector process of dimension d ($d = 0, 1, 2$ or 3), g, q and h are deterministic matrices of dimension $d \times k$, $\sigma_s(\xi) \geq 0$ and $a_s(\xi)$ are random field matrices of dimension $k \times m$ on $\mathbb{R}^3 \times \mathbb{R}$, W is an m-dimnsional white noise on $\mathbb{R}^3 \times \mathbb{R}$, L is an m-dimnsional nonnegative Lévy basis or exponential of a Lévy basis on $\mathbb{R}^3 \times \mathbb{R}$, and $A_t(x)$, $C_t(x)$ and $D_t(x)$ are (homogeneous) *ambit sets*, i.e. $A_t(x)$ is of the form $A_t(x) = A + (x, t)$ where

$$A = \left\{(\xi, s) : s \leq 0, c_s^- \leq \xi \leq c_s^+\right\}$$

for some functions $c_\cdot^-$ and $c_\cdot^+$ with $c_s^- \leq 0$ and $c_s^+ \geq 0$; and similarly for $C_t(x)$ and $D_t(x)$.

In this space-time setting the key questions (analogous to those discussed above) are substantially more intricate, major differences occurring already for the case of a one-dimensional space component. Here only a particular aspect of this will be discussed.

For simplicity we consider the case where the spatial dimension is 1 and $Y_t(x)$ is one-dimensional, i.e. $d = k = m = 1$.

0.9 Ambit processes

Now, let $\tau = \{\tau(w) : w \in \mathbb{R}\}$, with $\tau(w) = (\xi(w), s(w))$, be a smooth curve in $\mathbb{R} \times \mathbb{R}$ such that $s(w)$ is increasing in w and $s(\mathbb{R}) = \mathbb{R}$, and let

$$X_w = Y_{s(w)}(\xi(w))$$

with Y defined as in Section 0.8. The process $X = \{X_w\}_{w \in \mathbb{R}}$ is said to be an *ambit process*.

Under the specific assumptions made earlier

$$\begin{aligned} X_w &= \int_{A+\tau(w)} g(t-s, x-\xi)\sigma_s(\xi)\, W(\mathrm{d}\xi \mathrm{d}s) \\ &\quad + \int_{D+\tau(w)} q(t-s, x-\xi) a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s \end{aligned}$$

and we now consider the questions of whether the quadratic variation $[X_\cdot]$ exists, as the probability limit of the realised quadratic variation

$$[X_\delta]_w = \sum_{j=1}^{\lfloor w/\delta \rfloor} \left(X_{j\delta} - X_{(j-1)\delta}\right)^2,$$

and whether $[X_\cdot]_w = \int_0^w \sigma_{s(\phi)}^2(\xi(\phi))\, \mathrm{d}\phi$. A comprehensive treatment of these questions will not be attempted here, and we restrict the discussion to outlining a setting where the curve τ and the ambit set A are 'aligned' in a specified sense. A general formula is then available for the quadratic variation. Moreover, under certain conditions on g and A, X_w is representable as the difference $X_w = X_w^+ - X_w^-$ between two q-orthogonal semimartingales; however, such cases are not of prime interest in the context of turbulence and we shall not discuss them further here.

0.9.1 Alignment

Definition The curve τ and the ambit set A, with rectifiable and parametrised boundary $C = \{c(\gamma) : \gamma \in \Gamma\}$, are said to be *aligned* if the following conditions are satisfied. Let $c^{\perp}$ denote the transversal of $\dot{c}$, i.e. $c^{\perp} = (\dot{c}_2, -\dot{c}_1)$.

(i) For all w there exists a partition of C into two sets C_w^+ and C_w^- such that $\dot{\tau}(w) \cdot c^{\perp}(\gamma) \geq 0$ for all γ with $c(\gamma) \in C_w^+$ while $\dot{\tau}(w) \cdot c^{\perp}(\gamma) \leq 0$ for all γ with $c(\gamma) \in C_w^-$.

(ii) The subsets Γ_w^+ and Γ_w^- of Γ corresponding to C_w^+ and C_w^- are connected.

(iii) For all w the curve lengths of C_w^+ and C_w^- are positive.

The sets C_w^+ and C_w^- constitute the 'front' and the 'rear' of $A_{t(w)}(x(w))$ as $(x(w), t(w))$ moves along the curve τ.

Figures 0.1 and 0.2 illustrate a case of nonalignment and one of alignment, respectively.

0.9.2 QV under alignment

Suppose the curve τ and the ambit set A are aligned, and that A is convex and bounded. Then, under suitable conditions, the quadratic variation $[X.]$ of X exists as the limit in probability of the realised quadratic variation $[X_\delta]$ and

$$[X.]_w - [X.]_{w_0} = \int_{w_0}^{w} \int_C g^2(-c_1(\gamma), -c_2(\gamma)) \sigma^2(c(\gamma) + \tau(u)) \dot{c}^{\perp}(\gamma) \cdot \dot{\tau}(u) \mathrm{d}\gamma \mathrm{d}u.$$

In other words:

$$\mathrm{d}[X.]_w = \int_C g^2(-c_1(\gamma), -c_2(\gamma)) \sigma^2(c(\gamma) + \tau(u)) \dot{c}^{\perp}(\gamma) \cdot \dot{\tau}(w) \mathrm{d}\gamma \mathrm{d}w$$

which can be rewritten as

$$\frac{\mathrm{d}[X.]_w}{\mathrm{d}w} = \oint_{A+\tau(w)} g^2(\tau(w) - (\xi, s))^2 \sigma_s^2(\xi) \mathrm{d}\xi ds.$$

A detailed discussion of the pertinent conditions will be given elsewhere. Here we just mention that a conceptually important ingredient for the proof is the following pure analysis result (which is likely to be known but to which we have not been able to find a reference).

Let $m = 2$ and let $\tau(w)$ be a curve in $\mathbb{R}^2$ as before, and assume that τ and the boundary curve c of the ambit set A are both continuously differentiable. Furthermore, suppressing w in the notation $\tau(w)$, let

$$x_w = y_\tau = \int_{A+\tau} H(\tau, v)\, \mathrm{d}v$$

where the function $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be integrable on all sets $A + \tau$ and such that $H(t, x)$ is continuously differentiable with respect to t for almost all x (with respect to Lebesgue measure).

Proposition The differential of y_τ along τ is

$$\mathrm{d}y_\tau = \int_C H(\tau, c+\tau)\, \mathrm{d}c^\perp \cdot \mathrm{d}\tau + \int_{A+\tau} \mathrm{d}_\tau H(\tau, v)\, \mathrm{d}v \cdot \mathrm{d}\tau$$

where $\mathrm{d}c^\perp = (\mathrm{d}c_2, -\mathrm{d}c_1)$ is the transversal of $\mathrm{d}c$.

Sketch of proof. Suppose for simplicity that y_τ can be rewritten as

$$y_\tau = \int_{a+\tau_1(w)}^{b+\tau_1(w)} \int_{l(\xi)+\tau_2(w)}^{u(\xi)+\tau_2(w)} H(\tau, \xi, \eta)\, \mathrm{d}\eta \mathrm{d}\xi$$

Then, by ordinary rules of calculus, and using anticlockwise orientation for curvilinear integrals, we find

$$\begin{aligned}
\mathrm{d}y_\tau &= \int_{a+\tau_1(w)}^{b+\tau_1(w)} \mathrm{d} \int_{l(\xi)+\tau_2(w)}^{u(\xi)+\tau_2(w)} H(\tau(w), \xi, \eta)\, \mathrm{d}\eta \mathrm{d}\xi \\
&= \int_{a+\tau_1(w)}^{b+\tau_1(w)} H(\tau, \xi, u(\xi) + \tau_2(w))\, \mathrm{d}\tau_2 \mathrm{d}\xi \\
&\quad - \int_{a+\tau_1(w)}^{b+\tau_1(w)} H(\tau, \xi, l(\xi) + \tau_2(w))\, \mathrm{d}\tau_2 \mathrm{d}\xi \\
&\quad + \int_{a+\tau_1(w)}^{b+\tau_1(w)} \int_{l(\xi)+\tau_2(w)}^{u(\xi)+\tau_2(w)} \mathrm{d}_\tau H(\tau, \xi, \eta)\, \mathrm{d}\eta \mathrm{d}\xi \cdot \mathrm{d}\tau \\
&= - \int_{C+\tau} H(\tau, \xi, \eta)\, \mathrm{d}\xi \mathrm{d}\tau_2 + \int_{A+\tau} \mathrm{d}_\tau H(\tau, v)\, \mathrm{d}v \cdot \mathrm{d}\tau \\
&= \int_C H(\tau, c+\tau)\, \mathrm{d}c^\perp \cdot \mathrm{d}\tau + \int_{A+\tau} \mathrm{d}_\tau H(\tau, v)\, \mathrm{d}v \cdot \mathrm{d}\tau.
\end{aligned}$$

0.10 Conclusion

In the purely temporal setting, so far we have assumed that $\sigma \perp\!\!\!\perp B$. In joint work with José Manuel Corcuera and Mark Podolskij (Barndorff-Nielsen et al (2009)) this condition has been substantially weakened. This more refined analysis - which uses the theory of multipower variation and recent powerful results of Malliavin calculus due to Nualart, Peccati et al - has shown:

- In wide generality, $\overline{[Y_\delta]} \xrightarrow{p} \sigma^{2+}$

- Under certain conditions a feasible CLT for $\overline{[Y_\delta]}$ can be established.
- The results can be further extended to consistency and feasible CLTs for multipower variations, in particular for bipower variation.

Extensions of these results to the tempo-spatial regimes will be of key interest but the inclusion of a spatial component makes the issues considerably more challenging, as the discussion in Sections 0.8 and 0.9 will have indicated.

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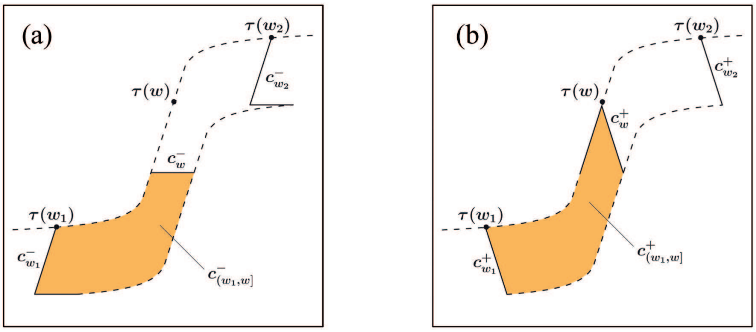

Figure 0.1 Illustration of the concept of alignment with a triangular ambit set. The curve τ and the triangular ambit set are not aligned.

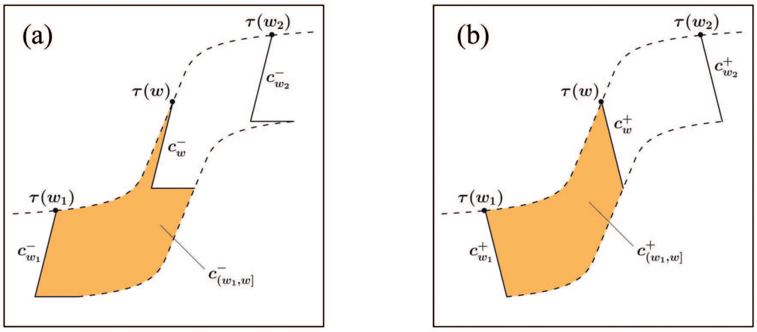

Figure 0.2 Illustration of the concept of alignment with a triangular ambit set. The curve τ and the triangular ambit set are aligned.

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