Stochastic PDEs with heavy-tailed noise

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Stochastic heat equation with Lévy noise

Stochastic heat equation:

$$\begin{aligned} \partial_t Y(t,x) &= \Delta Y(t,x) + \sigma(Y(t,x))\dot{\Lambda}(t,x) \\ Y(0,x) &= Y^0(x) \end{aligned}$$

- (t,x) time and space coordinate: $t \geq 0$, $x \in \mathbb{R}^d$
- $\dot{\Lambda}$ Lévy white noise on $\mathbb{R}_+ \times \mathbb{R}^d$
- σ a Lipschitz function
- Y^0 initial condition (in this talk $Y^0 \equiv 1$)

Approach by Walsh (1986)[1]:

$$Y(t,x) = Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(Y(s,y))\,\Lambda(\mathrm{d}s, \mathrm{d}y) \tag{SHE}$$

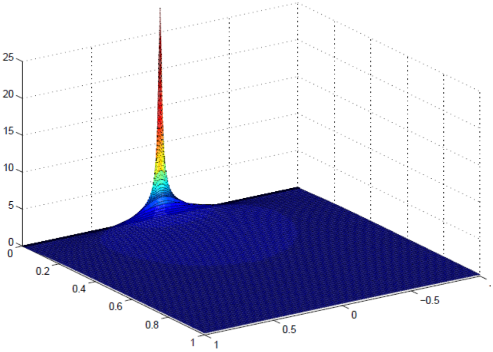
1. G is the heat kernel on $\mathbb{R}_+ \times \mathbb{R}^d$:

$$G(t,x) := \frac{\exp(-|x|^2/(4t))}{(4\pi t)^{d/2}} 1_{\{t>0\}}$$

2. $Y_0(t,x) := \int_{\mathbb{R}^d} G(t, x-y) Y^0(y)\,\mathrm{d}y$ (= 1 in this talk)

[1] Walsh, J.B. (1986): An introduction to stochastic partial differential equations. In Hennequin, P.L., editor, École d'Été de Probabilités de Saint Flour XIV - 1984, pages 265–439. Springer, Berlin.

Singularity and integrability



$$\int_0^T \int_{\mathbb{R}^d} G^p(t,x)\,\mathrm{d}(t,x) < \infty$$

$$\Longleftrightarrow$$

$$\boxed{0 < p < 1 + \frac{2}{d}}$$

Figure: The 1D heat kernel for $(t,x) \in [0,1] \times [-1,1]$

The Gaussian case

Λ **Gaussian white noise**

- Function-valued solutions only for $d = 1$
- A lot of work on generalizations and properties

$\rightarrow$ Dalang, Hairer, Khoshnevisan, Mueller, Walsh ...

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Our focus:

$$\boxed{\text{Noises with jumps}}$$

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Our focus:

Noises with jumps

In fact:

Noises with **heavy-tailed** jumps

The non-Gaussian case

Noise: Λ is a **homogeneous Lévy basis without Gaussian part**:

$$\Lambda(\mathrm{d}t,\mathrm{d}x) = b\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} z\mathbf{1}_{\{|z|\leq 1\}}\,(\mathfrak{p}-\mathfrak{q})(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{\mathbb{R}} z\mathbf{1}_{\{|z|>1\}}\,\mathfrak{p}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)$$

- $\mathfrak{p}$ Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ with intensity measure $\mathfrak{q}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\nu(\mathrm{d}z)$
- ν a **Lévy measure**

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Space of processes: define for $p \in (0, 2]$

$$B^p := \left\{ Y \text{ predictable} : \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p] < \infty \quad \forall T \in \mathbb{R}_+ \right\}$$

Existence and uniqueness

Theorem: Saint Loubert Bié (1998)[1]

If the Lévy measure ν satisfies

$$\int_{\mathbb{R}} |z|^p \, \nu(\mathrm{d}z) < \infty$$

for some $0 < p < 1 + 2/d$, then (SHE) has a unique solution in B^p.

[1] Saint Loubert Bié, E. (1998): Étude d'une EDPS conduite par un bruit poissonnien. *Probab. Theory Relat. Fields*, 111(2):287–321.

Problem

Crucial assumptions in previous work:

1. The Lévy measure satisfies

$$\int_{\mathbb{R}} |z|^p \, \nu(\mathrm{d}z) < \infty \quad \text{for some } p < 1 + \frac{2}{d},$$

Stable noises are excluded!

or

2. Noise has compact support in space (e.g. Balan (2014))

Heavy-tailed noise

Our goal:

$$\int_{|z|\le 1} |z|^p \,\nu(\mathrm{d}z) + \int_{|z|>1} |z|^q \,\nu(\mathrm{d}z) < \infty \quad \text{with} \quad \boxed{q < p}$$

Problems:

- Unlike SDEs with heavy-tailed noise, the classical stopping technique does not work
- p-th order moment estimates are infinite
- q-th order moment estimates exist and are finite, but do not produce contraction

Idea

Modified stopping strategy:

$\tau_N :=$ First time a jump of size $> N(1 + |x|^\eta)$ occurs at some $x \in \mathbb{R}^d$

Lemma

τ_N strictly positive and increasing to $+\infty$ if $\boxed{\eta > d/q}$

Idea

$$J(\phi)(t,x) := Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(\phi(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y).$$

New problem: p-th moments blow up in x in each iteration:

$$\mathbb{E}[|J(\phi)(t,x)|^p] \leq C_T \int_0^t \int_{\mathbb{R}^d} G^p(t-s,x-y)(1+\mathbb{E}[|\phi(s,y)|^p])(1+|y|^\eta)\,\mathrm{d}(s,y)$$

(**If $q = p$:**

$$\mathbb{E}[|J(\phi)(t,x)|^p] \leq C_T \int_0^t \int_{\mathbb{R}^d} G^p(t-s,x-y)(1+\mathbb{E}[|\phi(s,y)|^p])\,\mathrm{d}(s,y)\Bigg)$$

Idea

$$J(\phi)(t,x) := Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\sigma(\phi(s,y))\,\Lambda(\mathrm{d}s,\mathrm{d}y).$$

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$$\left(\textbf{If } q = p\textbf{:} \quad \mathbb{E}[|J(\phi)(t,x)|^p] \leq C_T \int_0^t \int_{\mathbb{R}^d} G^p(t-s,x-y)(1+\mathbb{E}[|\phi(s,y)|^p])\,\mathrm{d}(s,y) \right)$$

Still manageable, but one loses uniqueness (and global moment estimates)

Theorem[1]: Existence under heavy-tailed noise

If the Lévy measure ν satisfies

$$\int_{|z|\leq 1} |z|^p \, \nu(\mathrm{d}z) + \int_{|z|>1} |z|^q \, \nu(\mathrm{d}z) < \infty$$

with some

$$0 < p < 1 + \frac{2}{d} \quad \text{and} \quad q > \frac{p}{1 + (1 + \frac{2}{d} - p)}$$

then (SHE) has a solution Y such that for all $T, R \in \mathbb{R}_+$ and $N \in \mathbb{N}$

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}[|Y(t,x)|^p \mathbf{1}_{[\![0,\tau_N]\!]}(t)] < \infty$$

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[1] Chong, C. (2016): Stochastic PDEs with heavy-tailed noise. Preprint at arXiv:1602.00257 [math.PR].

Sketch of proof

Noise truncation: Let $h(x) := 1 + |x|^\eta$ and define

$$\Lambda^N(\mathrm{d}t, \mathrm{d}x) := b\,\mathrm{d}(t,x) + \int_{\mathbb{R}} z\mathbf{1}_{\{|z|\leq 1\}}(\mathfrak{p} - \mathfrak{q})(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z) + \int_{\mathbb{R}} z\mathbf{1}_{\{1<|z|\leq Nh(x)\}}\,\mathfrak{p}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z), \quad N \in \mathbb{N}.$$

Goal: Solution with $\Lambda = \Lambda^N$, then extend solution from τ_N to τ_{N+1}

Picard iteration:

$$Y^0(t,x) := 1, \quad Y^n(t,x) = J(Y^{n-1})(t,x).$$

$\longrightarrow$ Need moment estimates!

Sketch of proof

Moment estimates:

$$\mathbb{E}[|Y^n(t,x) - Y^{n-1}(t,x)|^p]$$

$$\leq C_T^n \int_0^t \int_{\mathbb{R}^d} \dots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} G^p(t-t_1, x-x_1) \dots G^p(t_{n-1}-t_n, x_{n-1}-x_n)$$
$$\times h(x_1)^{p-q} \dots h(x_n)^{p-q} \, \mathrm{d}(t_n, x_n) \dots \mathrm{d}(t_1, x_1)$$

$$\leq C_T^n \int_0^t \dots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} G^p(t-t_1, x_1) h(x-x_1)^{p-q} \dots$$
$$\times G^p(t_{n-1}-t_n, x_n) h(x-x_1-\dots-x_n)^{p-q} \, \mathrm{d}x_n \dots \mathrm{d}x_1 \, \mathrm{d}t_n \dots \mathrm{d}t_1.$$

Moment estimates:

$$\mathbb{E}[|Y^n(t,x) - Y^{n-1}(t,x)|^p]$$

$$\leq C_T^n \int_0^t \int_{\mathbb{R}^d} \dots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} G^p(t-t_1, x-x_1) \dots G^p(t_{n-1}-t_n, x_{n-1}-x_n) \times h(x_1)^{p-q} \dots h(x_n)^{p-q} \,\mathrm{d}(t_n, x_n) \dots \mathrm{d}(t_1, x_1)$$

$$\leq C_T^n \int_0^t \dots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} G^p(t-t_1, x_1) h(x-x_1)^{p-q} \dots \times G^p(t_{n-1}-t_n, x_n) h(x - x_1 - \dots - x_n)^{p-q} \,\mathrm{d}x_n \dots \mathrm{d}x_1 \,\mathrm{d}t_n \dots \mathrm{d}t_1.$$

The red integrals:

$$\int_{(\mathbb{R}^d)^n} G^p(t-t_1,x_1)\dots G^p(t_{n-1}-t_n,x_n)h(x-x_1-\dots-x_n)^{n(p-q)}\,\mathrm{d}(x_1,\dots,x_n)$$

Special property of heat kernel:

$$Ct^{\frac{d}{2}(p-1)}G^p(t,\cdot) = \text{ density of the } N(0,tI_d)\text{-distribution}$$

Thus: With $X \sim N(0,tI_d)$ we have

The red integrals

$$= C^n \prod_{j=1}^{n} (t_{j-1}-t_j)^{-\frac{d}{2}(p-1)} \mathbb{E}[h(x-X)^{n(p-q)}]$$

$$\le C^n \prod_{j=1}^{n} (t_{j-1}-t_j)^{-\frac{d}{2}(p-1)} \Gamma\left(\frac{1+n\eta(p-q)}{2}\right) \qquad \boxed{\text{for } |x| \le R}$$

Putting everything together:

$$\mathbb{E}[|Y^n(t,x) - Y^{n-1}(t,x)|^p]$$

$$\leq C_T^n \int_0^t \cdots \int_0^{t_{n-1}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G^p(t-t_1, x_1) h(x-x_1)^{p-q} \ldots$$

$$\times G^p(t_{n-1}-t_n, x_n) h(x - x_1 - \ldots - x_n)^{p-q} \, \mathrm{d}x_n \ldots \mathrm{d}x_1 \, \mathrm{d}t_n \ldots \mathrm{d}t_1.$$

$$\leq C_T^n \int_0^t \cdots \int_0^{t_{n-1}} \prod_{j=1}^n (t_{j-1} - t_j)^{-\frac{d}{2}(p-1)} \Gamma\left(\frac{1 + n\eta(p-q)}{2}\right) \mathrm{d}t_n \ldots \mathrm{d}t_1$$

$$= C_T^n \underbrace{\Gamma\left(\frac{1+n\eta(p-q)}{2}\right)}_{=\text{blow up due to stopping}} \Big/ \underbrace{\Gamma\left(1 + (1 - \tfrac{d}{2}(p-1))n\right)}_{=\text{size of iterated integrals}}$$

Summable in n under the stated hypotheses

□

Byproducts: Uniqueness?

No uniqueness in a "nice" space!

Byproducts: Uniqueness?

No uniqueness in a "nice" space!But:

Theorem

The constructed solution Y to (SHE) in the previous theorem is the unique solution in the space of predictable processes for which there exist a sequence of stopping times $(T_N)_{N\in\mathbb{N}}$ increasing to $+\infty$ a.s. and a process ϕ_0 such that for arbitrary $T, R \in \mathbb{R}_+$ and $K \in \mathbb{N}$ we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|\phi_0(t,x)|^p \mathbf{1}_{[\![0,T_K]\!]}(t)] < \infty,$$

and

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}[|(Y - J^{(n)}(\phi_0))(t,x)|^p \mathbf{1}_{[\![0,T_K]\!]}(t)] \to 0, \quad n \to \infty.$$

Byproducts: Stability under approximations

Theorem[1]

If Y^N is the solution to (SHE) with noise

$$\Lambda^N(\mathrm{d}t,\mathrm{d}x) = b\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} z\mathbf{1}_{\{1<|z|<N\}}\,\mathfrak{p}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) + \int_{\mathbb{R}} z\mathbf{1}_{\{|z|\leq 1\}}\,(\mathfrak{p}-\mathfrak{q})(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z),$$

then for all $T, R \in \mathbb{R}_+$ and $K \in \mathbb{N}$

$$\sup_{(t,x)\in[0,T]\times[-R,R]^d} \mathbb{E}[|Y(t,x)-Y^N(t,x)|^p \mathbf{1}_{[\![0,\tau_K]\!]}(t)] \to 0, \quad N\to\infty.$$

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Byproducts: Global moment bounds

Theorem[1]

If we additionally have that

$$|\sigma(x)| \leq C(1 + |x|^\gamma), \quad x \in \mathbb{R},$$

for some $C \in \mathbb{R}_+$ and $\gamma \in [0, q/p]$, then the constructed solution Y to (SHE) satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^q] < \infty, \quad T \geq 0.$$

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What did we gain?

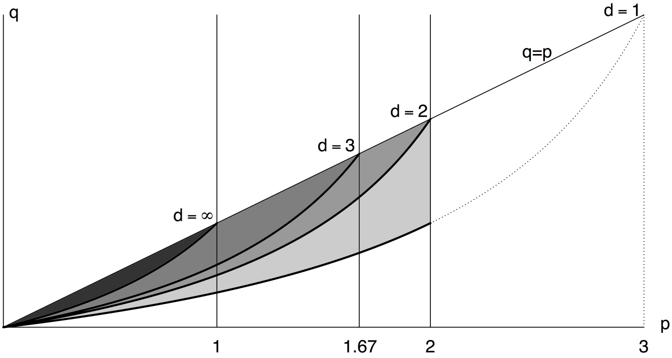

Figure: Constraints on p and q dependent on the dimension d; new area in grey

General parabolic SPDEs:

$$\partial_t Y(t,x) = \sum_{|\alpha|\leq 2m} c_\alpha(t,x)\partial^\alpha Y(t,x) + \sigma(Y(t,x))\dot{\Lambda}(t,x)$$

1. σ and $\dot{\Lambda}$ as before
2. $m \in \mathbb{N}$, c_α suitable continuous bounded functions

Result from PDE theory:

$$|G(t,x;s,y)| \leq C_T g(t-s, x-y),$$

$$g(t,x) = \frac{1}{t^{d/(2m)}} \exp\left(C \frac{|x|^{(2m)/(2m-1)}}{t^{1/(2m-1)}} \right)$$

Generalization

Theorem[1]

The previous theorems remain valid if the kernel G satisfies

$$|G(t,x)| \leq C_T t^{-\tau d/\rho} \mathrm{e}^{-C|x|^\rho/t^\tau} \mathbf{1}_{(0,\infty)}(t), \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

and

$$0 < p < 1 + \frac{\rho}{\tau d} \quad \text{and} \quad \frac{p}{1+\tau(1+\frac{\rho}{\tau d}-p)} < q \leq p.$$

In the parabolic SPDE case:

1. $\rho = (2m)/(2m-1)$
2. $\tau = 1/(2m-1)$

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Thank you very much!