Functional calculus and martingale representation formulas for on integer-valued random measures

Rama Cont & Pierre Blacque-Florentin

Dept of Mathematics
Imperial College London

Workshop on Ambit Processes and Related Topics, Aarhus 2016

Overview

Martingale representation theorem for jump processes

Let $J(dtdy)$ be an integer-valued random measure on $[0, T] \times \mathbb{R}_0^d$ with compensator $\mu(dtdy)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
The filtration $(\mathcal{F}_t)$ generated by J is said to have the predictable representation property if any $\mathcal{F}_t$-adapted square-integrable martingale is such that

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}_0^d} \psi(s, y)(J - \mu)(ds\ dy)$$

with $\psi : [0, T] \times \mathbb{R}_0^d \times \Omega \to \mathbb{R}_0^d$, $\mathcal{F}_t$-predictable.
The predictable representation property holds for Poisson random measures (Itô, Ikeda-Watanabe).
Conditions for measures with non-deterministic compensators are given in (Jacod 1987, Cohen 2013).

Martingale representation formulas

- Problem of finding an explicit representation appears in many applications like hedging, control of jump processes or BSDEs with jumps.
- Has been approached through Malliavin calculus for jump processes (Bismut 73, Jacod et al 1982, Lokka 05, Solé-Utzet-Vives 05,...) and Markovian techniques (Jacod-Méléard-Protter 00).
- In these results, ψ is represented in the form: $\psi(t,z) = {}^{p}E[D_{t,z}Y|\mathcal{F}_t]$, where D is an appropriate "Malliavin" derivative operator, for which many constructions have been proposed.

Outline

- We develop a calculus for functionals of integer-valued measures.
- For integer-valued random measure (IVRM) with the predictable representation property, we provide a pathwise construction of a 'stochastic derivative' operator, shown to be the adjoint of the compensated stochastic integral with respect to this IVRM.
- We provide an explicit version of the martingale representation formula for functionals of integer-valued random measures.
- These results extend the Functional Itô calculus to integer-valued random measures.

Canonical space of integer-valued random measures

Let $\mathbb{R}_0^d := \mathbb{R}^d - \{0\}$ and $\mathcal{M}_T := \mathcal{M}([0,T] \times \mathbb{R}_0^d)$ be space of integer-valued Radon measures on $[0,T] \times (\mathbb{R}_0^d)$:

$$j : \mathcal{B}([0,T] \times \mathbb{R}_0^d) \to \mathbb{N}$$

such that j is σ-finite and there exists a sequence of $(t_i, z_i) \in [0,T] \times \mathbb{R}_0^d$ such that

$$j(.) = \sum_{i=0}^{\infty} \delta_{(t_i, z_i)}(.)$$

$\mathcal{M}_2([0,T] \times \mathbb{R}_0^d)$ denotes the subset of measures with

$$\int_{[0,T]\times\mathbb{R}^d} \|z\|^2 j(dt\ dz) = \sum_{i \geq 0} \|z_i\|^2 < \infty$$

Non-anticipative functionals

Notation: for $j \in \mathcal{M}_T, t \in [0, T]$ denote j_t the restriction of j to $[0, t]$:

$$\forall A \in \mathcal{B}(\mathbb{R}_0^d), \mathbf{j_t}([0, T] \times A) = j([\mathbf{0}, \mathbf{t}] \times A).$$

and $\mathbf{j_{t-}}$ its restriction to $[0, t)$:

$$\forall A \in \mathcal{B}(\mathbb{R}_0^d), \mathbf{j_{t-}}([0, T] \times A) = j([\mathbf{0}, \mathbf{t}) \times A).$$

A map $F : [0, T] \times \mathcal{M}_T \to \mathbb{R}$ is said to be **non-anticipative** if

$$\forall j \in \mathcal{M}_T, \forall t \in [0, T], F(t, j) = F(t, j_t).$$

A map $F : [0, T] \times \mathcal{M}_T \to \mathbb{R}$ is said to be **predictable** if

$$\forall j \in \mathcal{M}_T, \forall t \in [0, T], F(t, j) = F(t, j_{t-}).$$

Integral functionals of integer-valued measures

Fundamental example: integral functionals
Let $F : [0, T] \times \mathcal{M}_T \to \mathbb{R}$ defined by

$$F(t,j) = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \psi(s,y)(j-\mu)(ds\ dy),$$

where $\psi : [0, T] \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a kernel with support bounded away from 0.
Then F is a non-anticipative functional.

A finite difference operator on functionals

For $z \in \mathbb{R}^d$, define

$$\nabla_{J,z} F(t,j) = F(t, j_{t-} + \delta_{(t,z)}) - F(t, j_{t-}) \tag{FD}$$

The operator

$$\begin{array}{rcl} \nabla_J F : [0,T] \times \mathcal{M}_T \times (\mathbb{R}^d - \{0\}) & \mapsto & \mathbb{R} \\ (t,j,z) & \to & \nabla_{J,z} F(t,j) \end{array}$$

maps non-anticipative functionals into predictable functionals.

Integral functionals of integer-valued measures

Proposition: integral functionals

Let $F_\psi : [0,T] \times \mathcal{M}_T \to \mathbb{R}$ defined by

$$F_\psi(t,j) = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \psi(s,y)(j-\mu)(ds\ dy),$$

where

- $\psi : [0,T] \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is a kernel with support bounded away from 0, and
- $\mu : \mathcal{B}([0,T] \times \mathbb{R}_0^d) \times \mathcal{M}([0,T] \times \mathbb{R}_0^d) \to \mathbb{R}^+$ is predictable in j and σ-finite.

Then F is a non-anticipative functional and $\nabla_J F_\psi = \psi$, i.e.

$$\forall (t,z) \in [0,T] \times \mathbb{R}_0^d, \quad \nabla_{J,z} F_\psi(t,j) = \psi(t,z).$$

Integer-valued random measures

We now consider $\mathcal{M}_T = \mathcal{M}([0,T] \times \mathbb{R}_0^d)$ endowed with the filtration $\mathbb{F}^0$ generated by the canonical process

$$\begin{array}{rcl} J : [0,T] \times \mathcal{B}([0,T] \times \mathbb{R}_0^d) \times \mathcal{M}_T & \to & \mathbb{R} \\ (t, A, j) & \to & j_t(A) = j([0,t] \cap A) \end{array}$$

We endow $(\Omega, \mathcal{F}_T)$ with a probability measure $\mathbb{P}$ such that the canonical process is an integer-valued random measure with $\mathbb{P}-$compensator $\mu(dt\ dz)$.
Let $\mathbb{F}$ be the $\mathbb{P}$-completed version of $\mathbb{F}^0_+$.

Multivariate point processes

Let us first consider the case of a multivariate point process for which $\mathbb{P}(J([0,T] \times \mathbb{R}_0^d) < \infty) = 1$. Then the the martingale representation property for $(J, \mathbb{F}, \mathbb{P})$ always holds (Jacod 1975):
a right continuous process Z is a $(\mathbb{P}, \mathbb{F})$l-ocal martingale if and only if there exists a predictable map $\psi : [0,T] \times \mathbb{R}_0^d \times \mathcal{M}_T \to \mathbb{R}$ such that

$$M(t) = M(0) + \int_0^t \int_E \psi(s,z)(J - \mu)(ds\, dz) = M(0) + \int_0^t \int_E \psi(s,z)\widetilde{J}(ds\, dz).$$

Martingale representation formula for point processes (Blacque-Florent, R.C 2015)

The integrand ψ has the following explicit representation:

$$\forall j \in \mathcal{M}_T, \quad \psi(t,z,j) = \nabla_J M(t,z,j) = M(j_{t-} + \delta_{(t,z)}) - M(j_{t-}) \qquad (1)$$

Extension to square-integrable IVRM

We now assume that the compensator μ of J satisfies

$$\mu(dsdy) = \nu(\{s\} \times dy)ds \text{ and } E^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}_0^d} (|z|^2 \wedge 1)\mu(dsdz)\right] < \infty,$$

and define
$\mathcal{L}^2_{\mathbb{P}}(\mu)$: { space of predictable random fields $\psi : [0, T] \times R^d \to \mathbb{R}$ such that

$$\|\psi\|^2_{\mathcal{L}^2_{\mathbb{P}}(\mu)} := E[\int_{[0,T]\times\mathbb{R}_0^d} |\psi(s,y)|^2 \mu(ds\ dy)] < \infty\}$$

$$\mathcal{I}^2_{\mathbb{P}}(\mu) :=$$

$$\{Y : [0, T] \times \Omega \to \mathbb{R} | Y(t) = \int_{[0,t]\times\mathbb{R}_0^d} \psi(s,y)(J-\mu)(dsdy), \psi \in \mathcal{L}^2_{\mathbb{P}}(\mu)\}$$

$$\|Y\|^2_{\mathcal{I}^2_{\mathbb{P}}(\mu)} := E[|Y(T)|^2]$$

Set $\mathcal{S}$ of cylindrical simple predictable fields

A predictable map $\psi : [0,T] \times \mathbb{R}^d \times \mathcal{M}([0,T] \times \mathbb{R}_0^d) \to \mathbb{R}^d$ belongs to $\mathcal{S}$ if has a representation

$$\psi(t, z, j_t) = \sum_{\substack{i=0 \\ k=1}}^{I,K} \psi_{ik}(j_{t_i}) 1_{(t_i, t_{i+1}]}(t) 1_{A_k}(z)$$

where

- $A_k \in \mathcal{B}([0,T] \times \mathbb{R}^d \backslash \{0\})$,$0 \notin \overline{A_k}$ and
- ψ_{ik} are bounded cylindrical functionals with support bounded away from zero.

Then for any $\psi \in \mathcal{S}$ and any $j \in \mathcal{M}_T$, the integral $\int \psi(s,z) j(dsdz)$ may be defined pathwise.

Stochastic operators

The compensated stochastic integral w.r.t J is defined as

$$I : \mathcal{L}^2_{\mathbb{P}}(\mu) \to \mathcal{I}^2_{\mathbb{P}}(\mu)$$
$$\psi \mapsto \int_0^{\cdot} \int_{\mathbb{R}^d_0} \psi(s,y)(J-\mu)(dsdy)$$

Proposition: The operator

$$\nabla_J : I(\mathcal{S}) \to \mathcal{L}^2_{\mathbb{P}}(\mu)$$

is defined (pathwise) on $I(\mathcal{S})$ and for $F(t,J) = \int_0^{\cdot} \int_{\mathbb{R}^d_0} \psi(s,y)(J-\mu)(dsdy)$, we have

$$\nabla_{J,z} F(t,J) = F(t, J_{t-} + \delta_{t,z}) - F(t, J_{t-}) = \psi(t,z)$$

Density of regular integral functionals

Proposition

The set $I(\mathcal{S})$ integral processes Y having a regular functional representation

$$Y(.) = F(., J) = \int_0^{.} \int_{\mathbb{R}^d \setminus \{0\}} \psi(s, y)(J - \mu)(dsdy) \tag{2}$$

with $\psi \in \mathcal{S}$, is dense in $\mathcal{I}^2_{\mathbb{P}}(\mu)$.

Then

$$\nabla_J : I(\mathcal{S}) \to \mathcal{S}$$

is defined pathwise and for $Y \in I(\mathcal{S})$ with the above representation,

$$\nabla_J Y(t, z) = \psi(t, z).$$

We now show that ∇_J is closable on $\overline{I(S)} = \mathcal{L}^2_{\mathbb{P}}(\mu)$.

∇_J as the adjoint of the stochastic integral

Theorem

The operator $\nabla_J : I(\mathcal{S}) \to \mathcal{L}^2_{\mathbb{P}}(\mu)$ is closable in $\mathcal{I}^2_{\mathbb{P}}(\mu)$, and is the adjoint of the stochastic integral in the sense of the following integration by parts.

$$
\begin{aligned}
< Y, I(\phi) >_{\mathcal{I}^2_{\mathbb{P}}(\mu)} &:= E\left[Y(T) \int_0^T \int_{\mathbb{R}^d \setminus \{0\}} \phi(s,y)(J-\mu)(dsdy) \right] \\
&= E\left[\int_0^T \int_{\mathbb{R}^d_0} \nabla_J Y(s,y) \phi(s,y) \mu(dsdy) \right] \\
&=: < \nabla_J Y, \phi >_{\mathcal{L}^2_{\mathbb{P}}(\mu)}
\end{aligned}
$$

Representation theorem for square-integrable martingales

The following result shows the link between the operator ∇_J and the martingale representation formula:

Martingale representation formula [R.C- Blacque-Florentin 2015]

Assume that the integer-valued random measure $(J, \mathbb{F}, \mathbb{P})$ has the predictable representation property. Then for any square integrable $(\mathbb{F}, \mathbb{P})$-martingale M,

$$M(t) = M(0) + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \nabla_J M(s,y)(J-\mu)(dsdy) \quad \mathbb{P}\text{-a.s.}$$

So ∇_J may be seen as a 'stochastic derivative' with respect to the compensated random measure $\tilde{J} = J - \mu$.

Comparison with Malliavin calculus on Poisson space

Comparing with representation formulae obtained through Malliavin calculus on Poisson space we obtain:

Proposition: Assume J is a Poisson random measure under $\mathbb{P}$ and let $\mathbb{D} : \mathbf{\Pi}^{1,2}(\mathbb{P}) \to L^2([0,T] \times \Omega)$ be the Malliavin derivative on Poisson space.Then

$$\forall H \ in \mathbf{\Pi}^{1,2}(\mathbb{P}), E[\mathbb{D}_{t,z} H | \mathcal{F}_t] = \nabla_J \ E[H|\mathcal{F}_t](t,z) \quad dt\mathbb{P} - a.e.$$

and the following diagram is commutative, in the sense of $dt \times d\mathbb{P}$ almost everywhere equality:

$$\begin{array}{ccc} \mathcal{I}^2(\mu) & \stackrel{\nabla_J}{\to} & \mathcal{L}^2(\mathbb{F}) \\ \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} & & \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} \\ \mathbf{\Pi}^{1,2} & \stackrel{\mathbb{D}}{\to} & L^2([0,T] \times \Omega) \end{array}$$

Note however that our construction works for more general integer-valued random measures and does not involve the Poisson /independence properties of the Poisson space.

Including the continuous component

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ constructed similarly as before, and $\mathbb{F}$ generated by an integer valued random measure J with compensator

$$\mu(dsdy) = \nu(\{s\} \times dy)ds \text{ and } \int_0^T \int_{\mathbb{R}_0^d} |z|^2 \mu(dsdz) < \infty, \mathbb{P}\text{-a.s.},$$

and a continuous martingale X, any square-integrable martingale writes, $\mathbb{P}$-a.s.

$$Y(T) = Y(0) + \int_0^T \nabla_X Y(s) dX(s) + \int_0^T \int_{\mathbb{R}_0^d} \nabla_J Y(s,z) \widetilde{J}(ds\ dz),$$

with $\nabla_X Y$ defined as follows.

Defining $\mathcal{S}_c$ as:

Set $\mathcal{S}_c$ of regular simple predictable processes

$\psi : [0,T] \times \times \mathcal{D}([0,T]) \times \mathcal{M}([0,T] \times \mathbb{R}_0^d) \to \mathbb{R}^d$ belongs to $\mathcal{S}_c$ if

- ϕ is predictable: $\psi(t,z,x,j) = \psi(t,z,x_{t-},j_{t-})$
- and

$$\phi(t,x_t,j_t) = \sum_{\substack{i=0\\k=1}}^{I} \phi_i(x_{t_i},j_{t_i}) 1_{(t_i,t_{i+1}]}(t)$$

with

$$\phi_i = g_i(x(\tau_1),\cdots,x(\tau_n),S_{\tau_1}^{\frac{1}{n}},\ldots,S_{\tau_n}^{\frac{1}{n}}),$$

$g_{ik} \in C_c^\infty(\mathbb{R}^{2n},\mathbb{R}^n)$ and $0 \le \tau_1 \le \tau_n \le t_i$,

$$S_t^\epsilon := j([0,t] \times (\epsilon,\infty)^d)$$

Define $\mathcal{L}^2_{\mathbb{P}}([X]) := \{$ space of predictable processes $\psi : [0, T] \times \Omega \to \mathbb{R}$ such that

$$\|\psi\|^2_{\mathcal{L}^2_{\mathbb{P}}([X])} := E[\int_{[0,T]\times\mathbb{R}^d_0} |\psi(s,y)|^2 [X](ds\ dy)] < \infty\}$$

and

$$\mathcal{I}^2_{\mathbb{P}}([X]) :=$$

$$\{Y : [0, T] \times \Omega \to \mathbb{R} | Y(t) = \int_{[0,t]\times\mathbb{R}^d_0} \phi(s) dX(s), \psi \in \mathcal{L}^2_{\mathbb{P}}([X])\}$$

$$\|Y\|^2_{\mathcal{I}^2_{\mathbb{P}}([X])} := E[|Y(T)|^2]$$

Defining:

$$\begin{aligned} I_X : & \mathcal{L}^2_{\mathbb{P}}([X]) \to \mathcal{I}^2_{\mathbb{P}}([X]) \\ & \phi \mapsto \int_0^{\cdot} \phi(s) dX(s), \end{aligned}$$

The operator

$$\begin{aligned} \nabla_x : & I_X(\mathcal{S}_c) \to \mathcal{I}^2_{\mathbb{P}}([X]) \\ & F(t, x_t, j_t) \mapsto \lim_{h \to 0} \frac{F(t, x_t + h 1_{[t,\infty)}, j_t) - F(t, x_t, j_t)}{h} \\ & = \phi(t) \end{aligned}$$

can be closed in $\mathcal{I}^2_{\mathbb{P}}([X])$ in the same fashion as in the jump case, and the closure is $\mathcal{I}^2_{\mathbb{P}}([X])$ itself.

Define the martingale-generating measure

$$M(ds\ dz) := 1_{\{z=0\}} dX(s) + z\widetilde{J}(ds\ dz),$$

Then the martingale representation formula rewrites, $\mathbb{P}$-a.s. as:

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} \nabla Y(s,z) M(ds\ dz) \quad \mathbb{P}\text{-a.s.}$$

where

$$\nabla Y(s,z) := \begin{cases} \nabla_X Y(s,y) \text{ if } z = 0 \\ \frac{\nabla_J Y(s,z)}{z} \text{ otherwise.} \end{cases}$$

The continuous component is the limit operator of the operator appearing in the jump case.

Example: representation of the supremum of a Lévy process

The representation formula for the supremum $\overline{X}$ of a Lévy process X

1. was proved by Shiryaev and Yor (2004) using Itô's formula.
2. was reproved more recently by Rémillard-Renaud(2011) using Malliavin calculus.

Main challenges in the functional Itô case:

1. Infinite variation: infinite variation, induced by a continuous component and/or an infinite jump activity destroys the pathwise characterisation of the quantities.
2. In case the Lévy process has a continuous component: the supremum is not a vertically differentiable functional.

$\hookrightarrow$ we need to truncate the jumps and smoothen the functional.

Functional approximation

Define the Lévy process

$$X(t) = X(0) + \mu t + \sigma W(t) + \int_0^t \int_{|z|<1} z\widetilde{J}(dsdz) + \int_0^t \int_{|z|\geq 1} zJ(dsdz)$$

and its approximation

$$X^n(t) = X(0)+\mu t+\sigma W(t)+\int_0^t \int_{(-1,-\frac{1}{n})\cup(\frac{1}{n},1)} z\widetilde{J}(dsdz)+\int_0^t \int_{|z|\geq 1} zJ(dsdz)$$

It can be shown that

$$E[\overline{X}(T)|\mathcal{F}_t] = \overline{X}(t) + \int_{\overline{X}(t)-X(t)}^{\infty} F_{T-t}(u)du,$$

with $F_{T-t}(u) = \mathbb{P}(\overline{X}(T-t) \leq u)$.

Furthermore, consider the approximation of the supremum functional,

$$L^a(f,t) = \frac{1}{a}\log(\int_0^t e^{af(s)}ds).$$

Define the approximation:

$$Y^{a,n}(t) = L^a(X^n,t) + \int_{L^a(X^n,t)-X^n(t)}^{\infty} F_{T-t}(u)du$$

Since $X^n \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} X$ and $L^a(f,T) \underset{a\to 0}{\longrightarrow} \sup_{s\in[0,T]} f(s)$, one can show:

$$\lim_{n\to\infty}\lim_{a\to\infty} E[|Y^{a,n}(T) - \overline{X}(T)|^2] = 0$$

We can now compute

$$\nabla_J Y^{a,n}(t,z) = \int_{L^a(X^n,t)-X^n(t)-z}^{L^a(X^n,t)-X^n(t)} F_{T-t}(u)du$$
$$\underset{\substack{a\to\infty \\ n\to\infty}}{\longrightarrow} \int_{\overline{X}(t)-X(t)-z}^{\overline{X}(t)-X(t)} F_{T-t}(u)du = \nabla_J \overline{X}(t,z)$$

and

$$\nabla_W Y^{a,n}(t) = \lim_{h\to 0} \frac{1}{h} \int_{L^a(X^n,t)-X^n(t)-\sigma h}^{L^a(X^n,t)-X^n(t)} F_{T-t}(u)du$$
$$= F_{T-t}(L^a(X^n,t) - X^n(t))$$
$$\underset{\substack{a\to\infty \\ n\to\infty}}{\longrightarrow} \sigma F_{T-t}(\overline{X}(T) - X(t)) = \nabla_W \overline{X}(t)$$

References

P. Blacque-Florentin and R. Cont (2015):
Functional calculus and martingale representation formula for integer-valued random measures
http://arxiv.org/abs/1508.00048