Limit theory for Lévy semistationary processes

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joint work with A. Basse-O'Connor and M. Podolskij

Conference on Ambit Fields and Related Topics

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AARHUS UNIVERSITY

In the last talk:

Limit theory for the power variation in the setting of infill asymptotics,

$$V(p)^n := \sum_{i=1}^{n} |\Delta_i^n X|^p, \qquad \Delta_i^n X := X_{\frac{i}{n}} - X_{\frac{i-1}{n}}.$$

Process $(X_t)_{t\in\mathbb{R}}$ a stationary increment moving average of the form

$$X_t = \int_{-\infty}^{t} \{g(t-s) - g_0(-s)\} dL_s$$

Limiting behavior of $V(p)^n$ is divided into three different regimes depending on

- β – the Blumenthal-Getoor index of the driving Lévy process L
- α – the power characterising the behavior of g at 0
- p – the power for the power variation.

In this talk:

1. Generalisation to Lévy semistationary processes:

$$X_t := \int_{-\infty}^{t} \{g(t-s) - g_0(-s)\}\sigma_s dL_s,$$

where σ is a predictable process.

2. Functional convergence
3. Applications and further generalisations.

Lévy semistationary (LSS) Processes, definition and assumptions

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$$X_t = \int_{-\infty}^{t} (g(t-s) - g_0(-s))\sigma_s \, \mathrm{d}L_s.$$

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- $(\sigma_t)_{t\in\mathbb{R}}$ is cádlág and predictable process, not necessarily independent of L.
- If σ is stationary and independent of L, then X is stationary.
- X generally not a semimartingale, nor an infinitely divisible process.

Motivation: Ambit fields and relative intermittency

$$X_t = \int_{-\infty}^{t} (g(t-s) - g_0(-s))\sigma_s \, \mathrm{d}L_s.$$

- LSS processes are an important purely temporal subclass of **ambit fields**, a class of stochastic processes introduced for modelling velocities in turbulent flows (Barndorff-Nielsen and Schmiegel 2005).

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- Typically, σ^2 is modelled as (exponential of an) ambit process, e.g. (Hedevang and Schmiegel 2013).

Limit theory for Brownian semistationary processes

(Barndorff-Nielsen, Corcuera, Podolskij 2009,2011): Limit theory for **BSS processes**:

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Denote $\tau_n^2 = \int_0^{1/n} g^2(x)dx + \int_0^{\infty}(g(1/n+x)-g(x))^2 \, \mathrm{d}x$.

Theorem 2.1

It holds that

$$n^{-1}\tau_n^{-p}\sum_{i=1}^{[tn]}|\Delta_i^n X|^p \xrightarrow{\mathbb{P}} \mathbb{E}[|\mathcal{N}(0,1)|^p] \int_0^t |\sigma_s|^p \, \mathrm{d}s.$$

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$\Rightarrow$ Consistent estimation of relative intermittency $(\int_0^t |\sigma_s|^2 ds)/(\int_0^1 |\sigma_s|^2 ds)$ possible, cf. (Barndorff-Nielsen, Pakkanen and Schmiegel 2015).

Pure jump LSS processes:

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where $(L_t)_{t\in\mathbb{R}}$ is a symmetric pure jump Lévy process with Lévy measure ν.

- $\beta \in [0,2)$: Blumenthal-Getoor index of L, defined as

$$\beta := \inf\left\{r \geq 0 : \int_{-1}^{1} |x|^r \nu(\mathrm{d}x) < \infty\right\}.$$

- $\alpha > 0$: Behavior of g at 0:

$$\lim_{t\downarrow 0} |g(t)|/t^{\alpha} = c_0 \in (0,\infty)$$

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The limiting behavior of $V(p)^n$ depends on α, β and p. We obtain three different regimes with different limits and convergence rates.

Theorem (Basse-O'Connor, H. and Podolskij)

(i): Assume that L is a SβS process with $\beta \in (0,2)$. If $\alpha \in (0, 1-1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p)^n \xrightarrow{\mathbb{P}} m_p \int_0^1 |\sigma_t|^p dt.$$

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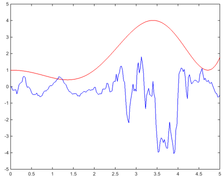
Proof: Bernstein's blocking technique



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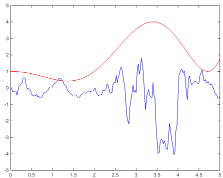

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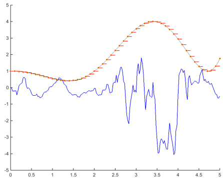
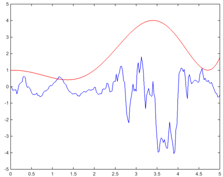


Step 1: $$Y_t = \int_{-\infty}^{t} g(t-s) dL_s$$

$$\widetilde{V}(p)^n = \sum_{i=1}^{n} |\sigma_{\frac{i-1}{n}} \Delta_i^n Y|^p$$

$$|n^{p(\alpha+1/\beta)-1}(V(p)_n - \widetilde{V}(p)^n)| \xrightarrow{\mathbb{P}} 0.$$



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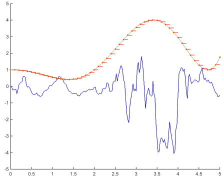

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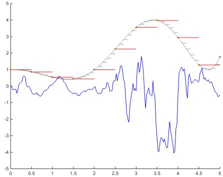
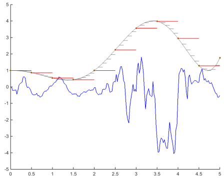


Step 2: Introduce second block size $1/l$.

$$\widetilde{V}(p)^{n,l} = \sum_{j=1}^{l} |\sigma_{\frac{j-1}{l}}|^p \left(\sum_{\frac{i}{n} \in \left[\frac{j-1}{l}, \frac{j}{l}\right)} |\Delta_i^n Y|^p \right)$$



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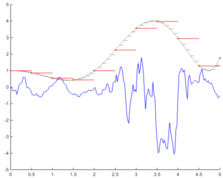
It holds for all $\varepsilon > 0$ that

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{P}(|n^{p(\alpha+1/\beta)-1}(\widetilde{V}(p)^{l,n} - \widetilde{V}(p)^n)| > \varepsilon) = 0$$



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Applying the limit theorem for constant σ we obtain

$$n^{p(\alpha+1/\beta)-1}\widetilde{V}(p)^{l,n} \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \sum_{j=1}^{l} |\sigma_{\frac{j-1}{l}}|^p \frac{m_p}{l} \quad \xrightarrow[l\to\infty]{\mathbb{P}} \quad m_p \int_0^1 |\sigma_t|^p dt.$$

Important ingredient: For asymptotic equivalence of $V(p)^n, \widetilde{V}(p)^n$ and $\widetilde{V}(p)^{l,n}$ we need the following isometry of the integral mapping.

Theorem (Kwapień, Woyczyński)

Let L be a symmetric β-stable Lévy process. There are positive constant c, C such that for all predictable F that are integrable w.r.t. L

$$c\mathbb{E}\Big[\int_{\mathbb{R}} |F_s|^\beta \, \mathrm{d}s\Big] \leq \Big\| \int_{\mathbb{R}} F_s \, \mathrm{d}L_s \Big\|_{\beta,\infty}^\beta \leq C\mathbb{E}\Big[\int_{\mathbb{R}} |F_s|^\beta \, \mathrm{d}s\Big],$$

where $\|\cdot\|_{\beta,\infty}$ denotes the weak $L^\beta(\Omega)$-norm.

The weak L^β-norm satisfy $\|X\|_{\beta'} \leq \|X\|_{\beta,\infty} \leq \|X\|_\beta$ for all $\beta' < \beta$.

Integration theory (Kwapień & Woyczyński, 1993), part 1

- Extension of the integration theory w.r.t. Lévy bases established in (Rajput & Rosiński 1989) towards predictable integrands

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- Decoupling inequalities approach

F is L-integrable (K & W) $\Leftrightarrow$ $F(\omega)$ is L-integrable (R & R), for almost all ω.
$\Leftrightarrow$ $\Phi_{L,0}(F) < \infty$, almost surely

Here, $\Phi_{L,0}$ is the functional

$$\Phi_{L,0}(F) := \int_{\mathbb{R}^2} |F_s x|^2 \wedge 1 \ \nu(\mathrm{d}x)\, \mathrm{d}s.$$

(Recall that L is a symmetric Lévy process)

Theorem (Basse-O'Connor, H. and Podolskij)

(ii) For $p \geq 1$, $\alpha > 1 - 1/(\beta \vee p)$ it holds that

$$n^{-1+p} V(p)^n \xrightarrow{\mathbb{P}} \int_0^1 |F_u|^p \,\mathrm{d}u$$

where

$$F_u = \int_{\infty}^{u} g'(u-s)\sigma_s \,\mathrm{d}L_s \quad \text{a.s.} \qquad \text{and} \qquad \int_0^1 |F_u|^p \,\mathrm{d}u < \infty \quad \text{a.s.}$$

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- For $\alpha > 1 - 1/(\beta \vee p)$, the sample paths of X are almost surely absolutely continuous with derivative F, cf. (Braverman and Samorodnitsky 1998).
- $\Rightarrow$ By mean value theorem: $n^{-1} \sum_{i=1}^n |n\Delta_i^n X|^p \approx n^{-1} \sum_{i=1}^n |F_{\frac{i-1}{n}}|^p$, for large n.

Theorem (Basse-O'Connor, H. and Podolskij)

Assume that $\alpha < 1 - 1/p$, $p > \beta$ and $p \geq 1$. We obtain the $\mathcal{F}$-stable convergence

$$n^{\alpha p} V(p)^n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m: T_m \in [0,1]} |\Delta L_{T_m} \sigma_{T_m}|^p Z_m.$$

Here, $(T_m)_{m \geq 1}$ is a sequence of stopping times exhausting the jumps of $(L_t)_{t \geq 0}$, and

$$Z_m = \sum_{l=0}^{\infty} |(l + U_m)^{\alpha} - (l + U_m - 1)_+^{\alpha}|^p,$$

where $(U_m)_{m \geq 1}$ is a sequence of independent and uniform $[0,1]$-distributed random variables, defined on an extension of the original probability space, independent of L and σ.

Note that Z_m is finite since $(\alpha - 1)p < -1$.

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- In which sense do we get convergence of $n^{\gamma} V(p)^n$ to a limiting process in $L^0(\Omega, \mathbb{D}([0,1]))$, where γ is the convergence rate established in the last section?

Theorem (Basse-O'Connor, H. and Podolskij)

(i'): Assume that L is a SβS process with $\beta \in (0,2)$. If $\alpha \in (0, 1-1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p)^n_t \stackrel{\text{u.c.p.}}{\Longrightarrow} m_p \int_0^t |\sigma_s|^p ds.$$

- $Z^n \stackrel{\text{u.c.p.}}{\Longrightarrow} Z$ ('uniformly on compacts in probability') if for all $C > 0$ and for all $\varepsilon > 0$

$$\mathbb{P}(\sup_{t\in[0,C]} |Z^n_t - Z_t| > \varepsilon) \to 0.$$

Theorem (Basse-O'Connor, H. and Podolskij)

(ii') For $p \geq 1$, $\alpha > 1 - 1/(\beta \vee p)$ it holds that

$$n^{-1+p} V(p)^n_t \stackrel{\text{u.c.p.}}{\Longrightarrow} \int_0^t |F_u|^p \, \mathrm{d}u$$

where $F_u = \int_{\infty}^u g'(u-s)\sigma_s \, \mathrm{d}L_s$.

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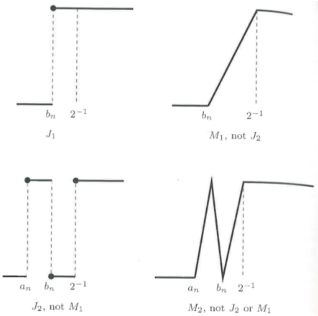

Figure 11.2. Four candidate sequences of functions $\{x_n : n \geq 1\}$ that might converge to $x \equiv I_{[1/2,1]}$ in $D([0,1],\mathbb{R})$, where $a_n = 2^{-1} - 2n^{-1}$ and $b_n = 2^{-1} - n^{-1}$.

Examples for convergence towards a function with a single jump in the different topologies.

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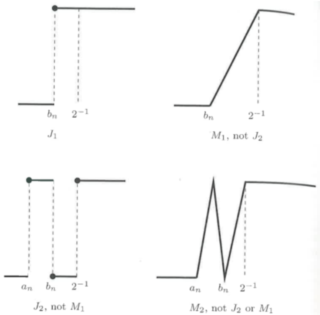

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Source: W.Whitt, Stochastic-Process Limits

(Avram & Taqqu 1998): Functional convergence of sums of moving averages w.r.t. M_1 but not J_1 topology.

Theorem (Basse-O'Connor, H. and Podolskij)

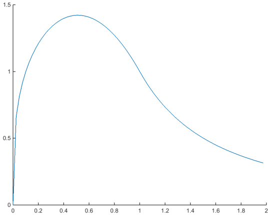
Assume that $\alpha < 1 - 1/p$, $p > \beta$ and $p \geq 1$. We obtain the functional $\mathcal{F}$-stable convergence

$$n^{\alpha p} V^n(p)_t \xrightarrow{\mathcal{L}_{M_1}-s} |c_0|^p \sum_{m: T_m \in [0,t]} |\Delta L_{T_m} \sigma_{T_m}|^p Z_m,$$

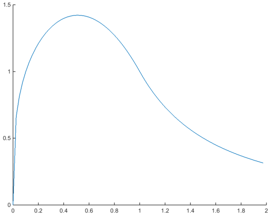
where

$$Z_m = \sum_{l=0}^{\infty} |(l + U_m)^{\alpha} - (l + U_m - 1)_+^{\alpha}|^p, \quad U_m \sim \mathcal{U}([0,1]).$$

The stable convergence in law does also hold with respect to the M_2 topology, but not with respect to the J_1 or J_2 topology.



$$g(x) \sim x^{\alpha}, \text{ as } x \to 0.$$



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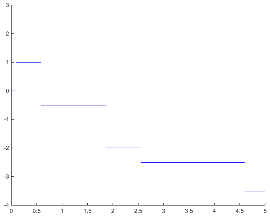
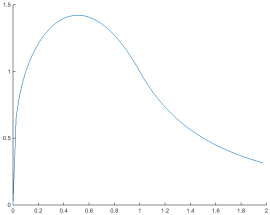


Lévy process L.



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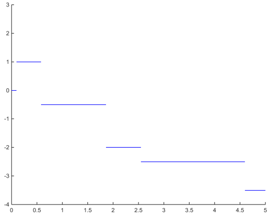

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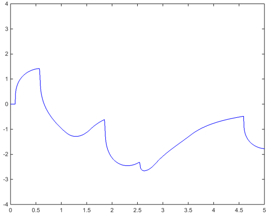
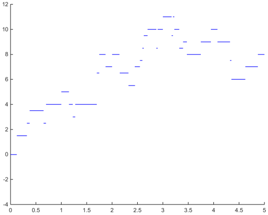


- $X_t = \int_{-\infty}^{t} g(t-s)\sigma_s dL_s$, $\sigma \equiv 1$.
- Jump times of Lévy process govern asymptotic behavior of $V(p)^n$.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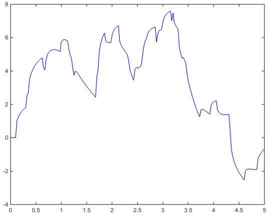
$\Rightarrow$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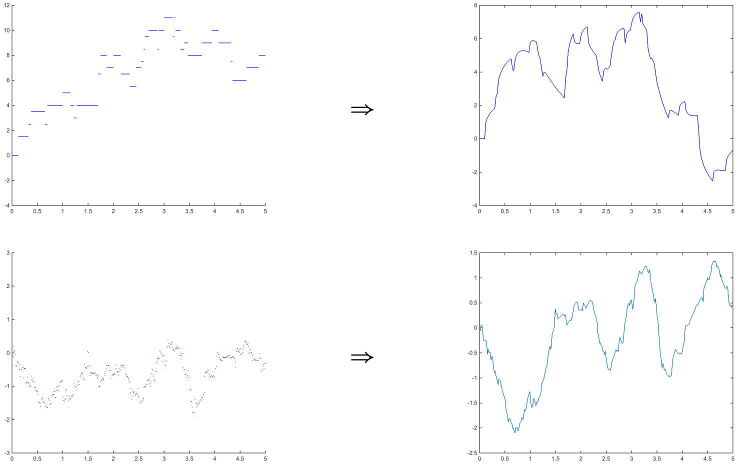
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⇒
8 6 4 2 0 -2 -4
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
3 2 1 0 -1 -2 -3
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
⇒
1.5 1 0.5 0 -0.5 -1 -1.5 -2 -2.5
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5

Let L be compound Poisson process with a jump at time $T \in ((i_0 - 1)/n, i_0/n]$, then

$$\Delta_{i_0}^n X \approx c_0(i_0/n - T)^{\alpha}\sigma_T \Delta L_T,$$
$$\Delta_i^n X \approx c_0((i/n - T)^{\alpha} - ((i-1)/n - T)^{\alpha})\sigma_T \Delta L_T.$$

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Lemma 1

For an absolutely continuous random variable Z with differentiable density we have the $\mathcal{F}$-stable convergence

$$\{nZ\} \xrightarrow{\mathcal{L}\text{-}s} U,$$

where $U \sim \mathcal{U}([0,1])$ is independent of Z. Here, $\{x\} = x - [x]$ denotes the fractional part of x.

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Since $nT \in ((i_0-1), i_0]$, we obtain

$$|\Delta_{i_0}^n X|^p \overset{d}{\approx} n^{-\alpha p}|c_0\sigma_T\Delta L_T|^p|U|^{\alpha p},$$
$$|\Delta_i^n X|^p \overset{d}{\approx} n^{-\alpha p}|c_0\sigma_T\Delta L_T|^p|(U+i-i_0)^\alpha - (U+i-i_0-1)^\alpha|^p.$$

Extension to general L

Idea: Let $a > 0$ and let $L^{>a}$ be the truncated Lévy process

$$L_t^{>a} - L_s^{>a} = \sum_{u \in (s,t]} \Delta L_u 1_{\{|\Delta L_u| > a\}},$$

and $L_t^{\le a} = L_t - L_t^{>a}$. Let

$$X_t^{>a} = \int_{-\infty}^t g(t-s)\sigma_s \, \mathrm{d}L_s^{>a}, \quad X_t^{\le a} = \int_{-\infty}^t g(t-s)\sigma_s \, \mathrm{d}L_s^{\le a}.$$

Claim: The error in the power variation caused by replacing X by $X^{>a}$ becomes negligible for $a \to 0$. More precisely, we show that $\limsup_{n\to\infty} \mathbb{P}[n^{p\alpha} V(X^{\le a}, p)_t^n > \varepsilon] \to 0$, as $a \to 0$ for all $\varepsilon > 0$.

Integration Theory (Kwapień & Woyczyński), part 2

Define for $p \geq 1$ and for a predictable process F the functional

$$\Phi_{L,p}(F) = \int_{\mathbb{R}^2} |F_s x|^2 1_{\{|F_s x| \leq 1\}} + |F_s x|^p 1_{\{|F_s x| > 1\}} \, \nu(\mathrm{d}x) \, \mathrm{d}s.$$

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Moreover, on the linear space of F with $\Phi_{L,p}(F) < \infty$ almost surely, introduce the random (quasi-)norm

$$\|F\|_{p,L} := \inf\{\lambda \geq 0 \, : \, \Phi_{p,L}(F/\lambda) \leq 1\}.$$

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Theorem (Kwapień & Woyczyński 1993)

There are positive constants c, C such that we obtain for all F with $\Phi_{L,p}(F) < \infty$

$$c\mathbb{E}\big[\|F\|^p_{p,L}\big] \leq \mathbb{E}\left[\left|\int_{\mathbb{R}} F_s \, \mathrm{d}L_s\right|^p\right] \leq C\mathbb{E}\big[\|F\|^p_{p,L}\big].$$

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For deterministic integrands the result was shown in (Rajput & Rosiński 1989).

Application: Estimation of α and β

Three regimes:

Thm (i): $\alpha < 1 - 1/\beta$, $p < \beta$. $n^{-1+p(\alpha+1/\beta)} V(p)^n$ converges

Thm (ii): $\alpha > 1 - 1/p$. $n^{p-1} V(p)^n$ converges

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Different convergence rates allow estimation of the parameters α and β :

$$S_{\alpha,\beta}(n,p) := -\frac{\log V(p)^n}{\log n}$$

$$S_{\alpha,\beta}(n,p) \xrightarrow{\mathbb{P}} S_{\alpha,\beta}(p) := \begin{cases} \alpha p : & \alpha < 1 - 1/p \text{ and } p > \beta \\ p(\alpha + 1/\beta) - 1 : & \alpha < 1 - 1/\beta \text{ and } p < \beta \\ p - 1 : & \alpha > 1 - 1/\max(p,\beta) \end{cases}$$

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$$(\hat{\alpha}_n, \hat{\beta}_n) := \operatorname{argmin}_{\alpha>0,\ \alpha+1/\beta\in(1/2,1)} \int_1^2 \left(S_{\alpha,\beta}(n,p) - S_{\alpha,\beta}(p)\right)^2 dp.$$

In the context of Theorem (i), that is for β-stable driving Lévy process, and $\alpha < 1 - 1/\beta$, $p < \beta$, we obtain

$$\frac{\sum_{i=1}^{[tn]} |\Delta_i^n X|^p}{\sum_{i=1}^{n} |\Delta_i^n X|^p} \xrightarrow{\mathbb{P}} \frac{\int_0^t |\sigma_s|^p \, \mathrm{d}s}{\int_0^1 |\sigma_s|^p \, \mathrm{d}s}, \quad t \in (0,1).$$

The right hand side is the **relative intermittency** and plays an important role in turbulence applications.

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We allow for kth order increments:

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}, \qquad i \geq k.$$

The limiting behavior of $V_n(X, p; k)$ depends on the interplay between α, β, p and k.

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