Asymptotic theory for large sample autocovariances matrices with heavy-tailed entries

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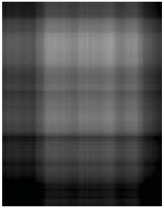
Motivation and Applications I

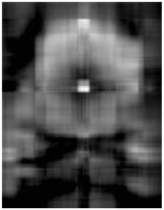
- PCA
- Image compression: low-dimensional approximation based on SVD may be sufficient.

color
600 x 465 x 3

grayscale
600 x 465 = 484k



1 dimension
600 + 465 + 1 = 1k



5 dimensions
5(600 + 465 + 1) = 5k

Game Plan

1. Classical Random Matrix Theory (RMT)
2. Largest, smallest eigenvalue under various moment assumptions
3. Heavy-tailed entries: the iid case
4. Heavy-tailed entries: with dependence
5. Sample correlation matrices

Stieltjes transform

For any sequence $(\mathbf{A}_n)$ of $p \times p$ matrices with only real eigenvalues $\lambda_1(\mathbf{A}_n), \ldots, \lambda_p(\mathbf{A}_n)$ the *empirical spectral distribution* is

$$F_{\mathbf{A}_n}(x) = \frac{1}{p} \sum_{i=1}^{p} \mathbb{1}_{\{\lambda_i(\mathbf{A}_n) \le x\}}, \qquad x \in \mathbb{R}, \quad n \ge 1.$$

In random matrix theory a lot of attention has been given to the problem of finding a distribution function F such that $F_{\mathbf{A}_n} \to F$ at all continuity points of F.
Yesterday: Steen Thorbjørnsen's talk: semicircle law
The *Stieltjes transform* of the empirical spectral distribution $F_{\mathbf{A}}$ is

$$s_{\mathbf{A}}(z) = \int \frac{1}{x - z} \, \mathrm{d}F_{\mathbf{A}}(x) = \frac{1}{p} \operatorname{tr}(\mathbf{A} - z\mathbf{I})^{-1},$$

where $z = u + \mathrm{i}v \in \mathbb{C}^+$, the complex numbers with positive imaginary part. The convergence $d(F_{\mathbf{A}_n}, F) \to 0$ is equivalent to $s_{F_{\mathbf{A}_n}}(z) \to s_F(z)$ for all $z \in \mathbb{C}^+$.

Setup & Objective

Data matrix: a $p \times n$ matrix $\boldsymbol{X} = \boldsymbol{X}_n$ consisting of n observations of a p-dimensional time series, i.e.

$$\boldsymbol{X} = (X_{it})_{i=1,\dots,p;t=1,\dots,n}.$$

We are interested in the non-normalized $p \times p$ *sample covariance matrix* $\boldsymbol{X}\boldsymbol{X}'$ and its *ordered eigenvalues*

$$\lambda_{(1)} \geq \lambda_{(2)} \geq \cdots \geq \lambda_{(p)}.$$

The Marčenko–Pastur Law

Let $\boldsymbol{X}$ have iid, real-valued, centered entries with variance 1.
Assume $p/n \to \gamma \in (0,1]$.
The empirical spectral distribution $F_{\frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}$ converges to a deterministic distribution with density supported on $[x_-, x_+]$, where $x_- = (1-\sqrt{\gamma})^2$ and $x_+ = (1+\sqrt{\gamma})^2$, given by

$$\frac{\sqrt{(x-x_-)(x_+-x)}}{2\pi x\gamma} \mathbb{1}_{[x_-,x_+]}(x).$$

Direct implications from the Marčenko–Pastur law:

$$\limsup_{n\to\infty} \frac{\lambda_{(p)}}{n} \le (1-\sqrt{\gamma})^2 \le (1+\sqrt{\gamma})^2 \le \liminf_{n\to\infty} \frac{\lambda_{(1)}}{n}.$$

Limiting spectral distribution under heavy tails

Assumption: regular variation of iid entries, infinite second moment.

Then $(F_{a_{n+p}^{-2}\boldsymbol{X}\boldsymbol{X}'})$ converges weakly with probability one to a deterministic probability measure whose density ρ_α^γ satisfies

$$\rho_\alpha^\gamma(x)x^{1+\alpha/2} \to \frac{\alpha\gamma}{2(1+\gamma)}, \qquad x \to \infty\,,$$

see [Belinschi et al., 2009, Theorem 1.10] and [Ben Arous and Guionnet, 2008, Theorem 1.6].

Extreme eigenvalues + light tails

Assume that $\boldsymbol{X}$ has iid centered entries with unit variance.

Finite fourth moment of X_{ij}.

If $\lim_{n\to\infty} p/n = \gamma \in (0,\infty)$, then

$$\frac{1}{n}\lambda_{(1)} \to (1+\sqrt{\gamma})^2 \quad \text{a.s.}$$

In particular, if $\boldsymbol{X}$ has iid standard normal entries [Johnstone, 2001] showed that

$$n^{2/3}\frac{(\sqrt{\gamma})^{1/3}}{(1+\sqrt{\gamma})^{4/3}}\Big(\frac{\lambda_{(1)}}{n} - \big(1+\sqrt{\tfrac{p}{n}}\big)^2\Big) \stackrel{d}{\to} \text{Tracy–Widom distr.},$$

which is a generic distribution in Random Matrix Theory.

Recently [Tikhomirov, 2015] proved $n^{-1}\lambda_{(p)} \to (1-\sqrt{\gamma})^2$ if $\boldsymbol{X}$ has unit variance.

Four Moment Theorem, [Tao and Vu, 2010]

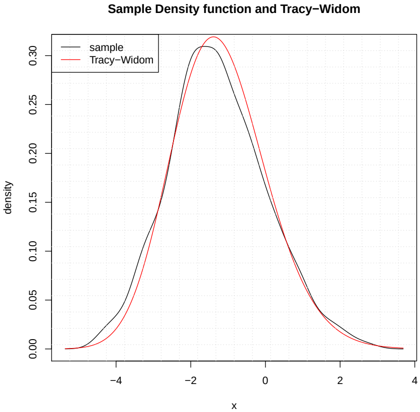


Figure : Entry distribution: $\mathbb{P}(X = \sqrt{3}) = \mathbb{P}(X = -\sqrt{3}) = 1/6$, $\mathbb{P}(X = 0) = 2/3$. Note $\mathbb{E}X = 0$, $\mathbb{E}[X^2] = 1$, $\mathbb{E}[X^3] = 0$ and $\mathbb{E}[X^4] = 3$, i.e., the first 4 moments of X match those of the standard normal distribution .

Heavy tails, iid case

Infinite fourth moment of X_{ij}.

If $\boldsymbol{X}$ is an $n \times n$ matrix with iid entries, [Bai et al., 1988] showed that

$$\limsup_{n\to\infty} \frac{\lambda_{(1)}}{n} = \infty \quad \text{a.s.}$$

We need a stronger normalization than n.

Infinite fourth moment, $p \to \infty$

- Assume the iid entries X_{ij} are **regularly varying** with index $\alpha \in (0,4)$, i.e. $\mathbb{P}(|X| > x) = x^{-\alpha} L(x)$ as $x \to \infty$, and

  $$\mathbb{P}(X > x) = q x^{-\alpha} L(x) \quad \text{and} \quad \mathbb{P}(X < -x) = (1-q) x^{-\alpha} L(x)$$

  for some $q \in [0,1]$.
- **Normalizing sequence** (a_{np}^2): (a_n) such that

  $$n\mathbb{P}(|X_{11}| > a_n x) \to x^{-\alpha}, \quad \text{as } n \to \infty \text{ for } x > 0.$$

  Then $a_{np} = (np)^{1/\alpha} \ell(np)$.
  We have

  $$\lim_{n\to\infty} \frac{a_{nn}^2}{n} = \infty.$$
- **Growth condition**: $p = n^\beta L_1(n) \to \infty$ for $\beta \in [0,1]$.
  Since $\boldsymbol{X}\boldsymbol{X}'$ and $\boldsymbol{X}'\boldsymbol{X}$ have the same non-zero eigenvalues it is enough to consider $\beta \in [0,1]$.

More notation

Let

$$D_i = (\boldsymbol{X}\boldsymbol{X}')_{ii} = \sum_{t=1}^{n} X_{it}^2$$

and denote by $D_{(i)}$ their order statistics.
We denote the order statistics of the random variables $X_{it}^2, i = 1, \dots, p; \; t = 1, \dots, n$ by

$$X_{(1),np}^2 \geq X_{(2),np}^2 \geq \dots \geq X_{(np),np}^2.$$

Main Result in the heavy-tailed iid case

Theorem

Consider a $p \times n$-dimensional matrix $\boldsymbol{X}$ with iid regularly varying entries with index $\alpha \in (0,4)$. We assume $\mathbb{E}[X] = 0$ for $\alpha \geq 2$ Then the following statements hold:

1. If $\beta \in [0,1]$, then

$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - D_{(i)}\right| \stackrel{\mathbb{P}}{\rightarrow} 0\,.$$

2. If $\beta \in ((\alpha/2-1)_+, 1]$, then

$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - X^2_{(i),np}\right| \stackrel{\mathbb{P}}{\rightarrow} 0\,.$$

Reduction to diagonal

Diagonal

Assume that $\boldsymbol{X} = \boldsymbol{X}_n$ has iid entries satisfying the regular variation condition for some $\alpha \in (0,4)$. If $\mathbb{E}[|X|] < \infty$ we also suppose that $\mathbb{E}[X] = 0$. Then for any sequence (p_n) satisfying $p_n = n^\beta \ell(n)$ with $\beta \in [0,1]$ we have

$$a_{np}^{-2} \|\boldsymbol{X}\boldsymbol{X}' - \operatorname{diag}(\boldsymbol{X}\boldsymbol{X}')\|_2 \stackrel{\mathbb{P}}{\to} 0\,, \qquad n \to \infty\,,$$

where $\|\cdot\|_2$ denotes the spectral norm.

$$(\boldsymbol{X}\boldsymbol{X}')_{ij} = \sum_{t=1}^n X_{it} X_{jt}.$$

Weyl

For any symmetric $p \times p$ matrices A, B, by *Weyl's inequality*

$$\max_{i=1,\dots,p} \left|\lambda_{(i)}(A+B) - \lambda_{(i)}(A)\right| \le \|B\|_2 \,.$$

If we now choose $A+B = \boldsymbol{X}\boldsymbol{X}'$ and $A = \mathrm{diag}(\boldsymbol{X}\boldsymbol{X}')$ we obtain the following result:

$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - \lambda_{(i)}(\mathrm{diag}(\boldsymbol{X}\boldsymbol{X}'))\right| \stackrel{\mathbb{P}}{\to} 0 \,, \quad n \to \infty \,.$$

Thus the problem of deriving limit theory for $(\lambda_{(i)})$ has been reduced to limit theory for the order statistics of the eigenvalues of $\mathrm{diag}(\boldsymbol{X}\boldsymbol{X}')$.

Eigenvectors

- $\mathrm{diag}(\boldsymbol{X}\boldsymbol{X}')$.
- Eigenvectors are canonical basisvectors e_j.

Eigenvectors

Assume the conditions of the Theorem and let $\beta \in [0,1]$. Then for any fixed $k \geq 1$,

$$\|v_k - e_{L_k}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0\,, \quad n \to \infty\,.$$

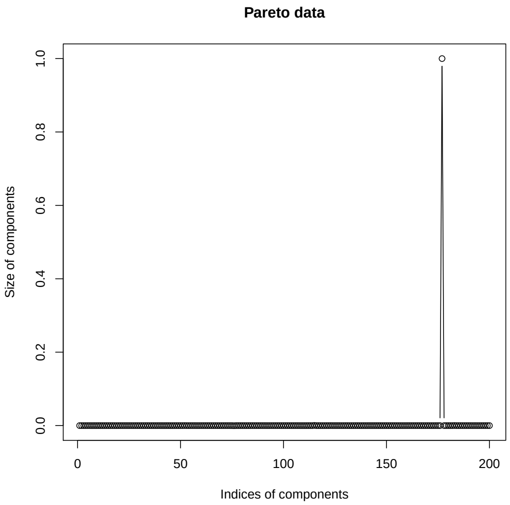


Figure : Eigenvectors: In the case of Pareto tails, $\max_{i=1,\dots,p} |v_{1,i}| = 1 - 10^{-5}$ The values used in the simulations are $p = 200$, $n = 1000$, $\alpha = 0.8$.

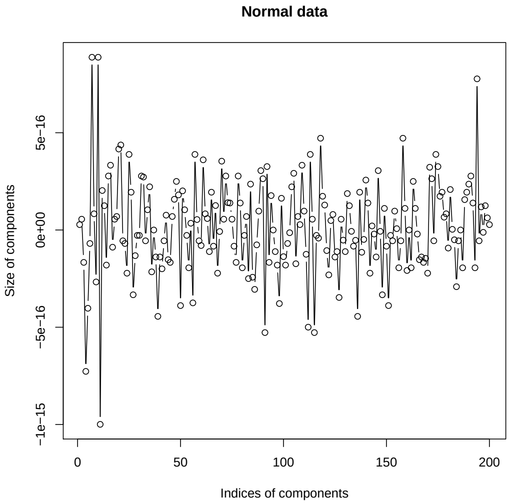


Figure : Eigenvectors: Standard normal. The values used in the simulations are $p = 200$, $n = 1000$.

Main Result in the heavy-tailed iid case

Theorem

Consider a $p \times n$-dimensional matrix $\boldsymbol{X}$ with iid regularly varying entries with index $\alpha \in (0,4)$. We assume $\mathbb{E}[X] = 0$ for $\alpha \geq 2$ Then the following statements hold:

1. If $\beta \in [0,1]$, then
$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - D_{(i)}\right| \xrightarrow{\mathbb{P}} 0\,.$$
2. If $\beta \in ((\alpha/2-1)_+, 1]$, then
$$a_{np}^{-2} \max_{i=1,\dots,p} \left|\lambda_{(i)} - X^2_{(i),np}\right| \xrightarrow{\mathbb{P}} 0\,.$$

For $\beta = 1$ this result was proven in [Auffinger et al., 2009]. The study of Hermitean matrices with power-law entries was started by [Soshnikov, 2004, Soshnikov, 2006].

Illustration

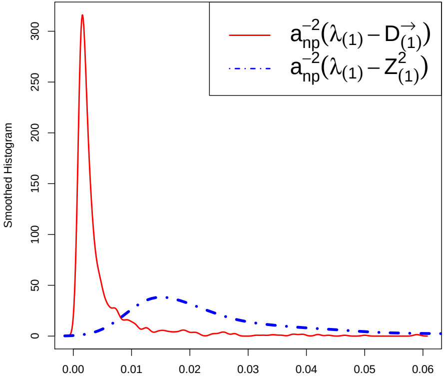


Figure : Smoothed histograms of the approximation errors for the normalized eigenvalues $(a_{np}^{-2}\lambda_{(i)})$ for entries X_{it} with $\alpha = 1.6$, $\beta = 1$, $n = 1000$ and $p = 200$.

Applications

Then

$$N_n = \sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{d} \sum_{i=1}^{\infty} \varepsilon_{\Gamma_i^{-2/\alpha}} = N.$$

The limit is a PRM on $(0,\infty)$ with mean measure $\mu(x,\infty) = x^{-\alpha/2}, x > 0$, and

$$\Gamma_i = E_1 + \cdots + E_i, \quad (E_i) \text{ iid standard exponential.}$$

For fixed $k \geq 1$:

$$\lim_{n\to\infty} \mathbb{P}(a_{np}^{-2}\lambda_{(k)} \leq x) = \lim_{n\to\infty} \mathbb{P}(N_n(x,\infty) < k) = \mathbb{P}(N(x,\infty) < k)$$
$$= \sum_{s=0}^{k-1} \frac{\big(\mu(x,\infty)\big)^s}{s!} \, \mathrm{e}^{-\mu(x,\infty)}, \quad x > 0.$$

In particular,

$$\frac{\lambda_{(1)}}{a_{np}^2} \xrightarrow{d} \Gamma_1^{-\alpha/2}, \qquad n \to \infty,$$

where the limit has a *Fréchet distribution* with parameter $\alpha/2$.

Applications

- Mapping theorem: For fixed $k \in \mathbb{N}$

$$a_{np}^{-2}(\lambda_{(1)}, \ldots, \lambda_{(k)}) \xrightarrow{d} (\Gamma_1^{-2/\alpha}, \ldots, \Gamma_k^{-2/\alpha}) = Y_k,$$

$$a_{np}^{-2}\big(\lambda_{(1)} - (p \vee n)\mathbb{E}[X^2], \ldots, \lambda_{(k)} - (p \vee n)\mathbb{E}[X^2]\big) \xrightarrow{d} Y_k\,.$$

- We also have

$$\Big(\frac{\lambda_{(2)}}{\lambda_{(1)}}, \ldots, \frac{\lambda_{(k)}}{\lambda_{(k-1)}}\Big) \xrightarrow{d} \Big(\Big(\frac{\Gamma_1}{\Gamma_2}\Big)^{2/\alpha}, \ldots, \Big(\frac{\Gamma_{k-1}}{\Gamma_k}\Big)^{2/\alpha}\Big).$$

- Law of large numbers:

$$\frac{\lambda_{(k+1)}}{\lambda_{(k)}} \xrightarrow{\mathbb{P}} 1, \quad k \to \infty.$$

Application: S&P 500 index

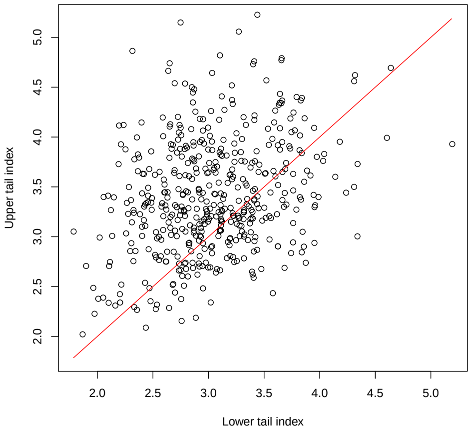


Figure : Estimated tail indices of stock returns in the S&P 500 index.

Application: S&P 500 index

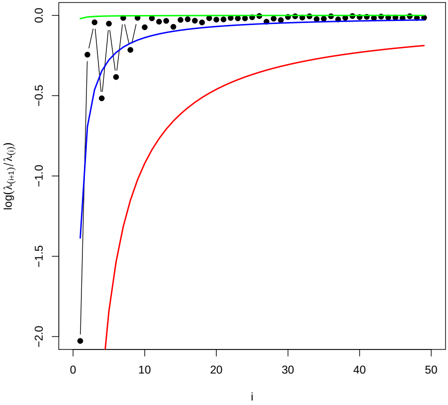


Figure : The logarithms of the ratios $\lambda_{(i+1)}/\lambda_{(i)}$ for the S&P 500 series after rank transform. We also show the 1, 50 and 99% quantiles (bottom, middle, top lines, respectively) of the variables $\log((\Gamma_i/\Gamma_{i+1})^2)$.

Application: S&P 500 index, original data (no rank transform)

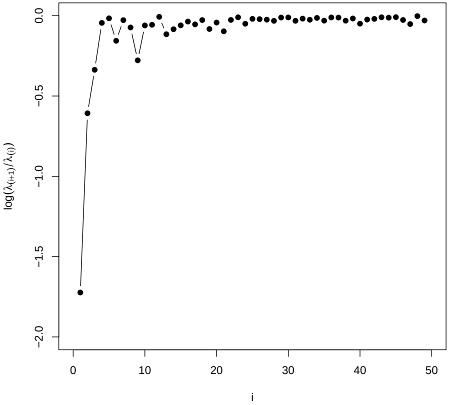


Figure : The ratios $(\lambda_{(i)}/\lambda_{(i+1)})$ for the original (non-rank transformed) S&P 500 log-return data.

Let (Z_{it}) be a field of regularly varying random variables.

- **Stochastic volatility model**:

$$X_{it} = Z_{it}\,\sigma_{it}^{(n)}\,.$$

- **Generate covariance structure** A:

$$\boldsymbol{X} = A^{1/2}\mathbf{Z}\,.$$

- **Dependence among rows and columns**:

$$X_{it} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} h_{kl} Z_{i-k,t-l}$$

with some constants h_{kl}.

Asymptotic theory for large sample covariance matrices

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Abstract

In risk management an appropriate assessment of the dependence structure of multivariate data plays a crucial role for the trustworthiness of the obtained results. The case of *heavy-tailed components* is of particular interest.
We consider asymptotic properties of sample covariance matrices for such time series, where both the dimension and the sample size tend to infinity simultaneously.

Known Results

If the rows of $\boldsymbol{X}$ are independent and identically distributed strictly stationary ergodic time series, then for fixed p we have $\frac{1}{n}\boldsymbol{X}\boldsymbol{X}' \xrightarrow{\text{a.s.}} \boldsymbol{I}_p$.
In particular, if $\boldsymbol{X}$ has iid standard normal entries Johnstone (2001) showed that for $p, n \to \infty$ with $p/n \to \gamma > 0$,

$$n^{2/3}\frac{(\sqrt{\gamma})^{1/3}}{(1+\sqrt{\gamma})^{4/3}}\Big(\frac{\lambda_{(1)}}{n} - (1+\sqrt{\tfrac{p}{n}})^2\Big) \xrightarrow{d} \text{TW},$$

a Tracy-Widom distribution.
Let us now assume that the entries of $\boldsymbol{X}$ are still iid but with *infinite fourth moment* (heavy tails). Since $\limsup \lambda_{(1)}/n = \infty$ a.s. a much stronger normalization of $\boldsymbol{X}\boldsymbol{X}'$ is required.

Our Model

Suppose $\boldsymbol{X} = (X_{it})_{i=1,\dots,p; t=1,\dots,n}$ with

$$X_{it} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} h_{kl} Z_{i-k,t-l}$$

and **regularly varying** iid noise (Z_{it}) with index $\alpha \in (0,4)$ (infinite fourth moment), i.e. there exists a normalizing sequence (a_n) such that

$$n\mathbb{P}(|Z| > a_n x) \to x^{-\alpha}, \quad \text{as } n \to \infty \text{ for } x > 0,$$

and a tail balance condition holds. If Z is regularly varying with index α, then moments above the αth do not exist.
Moreover we impose a summability condition on the double array of real numbers (h_{kl}) and a very general growth condition on $p = p_n \to \infty$.

Setup & Objective

Data matrix: a $p \times n$ matrix $\boldsymbol{X}$ consisting of n observations of a p-dimensional time series, i.e.

$$\boldsymbol{X} = (X_{it})_{i=1,\dots,p; t=1,\dots,n}.$$

We are interested in the non-normalized $p \times p$ *sample covariance matrix* $\boldsymbol{X}\boldsymbol{X}'$ and its *ordered eigenvalues*

$$\lambda_{(1)} \ge \lambda_{(2)} \ge \dots \ge \lambda_{(p)}.$$

Main Result

The order statistics $D_{(i)}$ of the iid sequence $D_s = \sum_{t=1}^{n} Z_{st}^2$ and the ordered eigenvalues $v_{(j)}$ of the matrix M given by $M_{ij} = \sum_{\ell=0}^{\infty} h_{i\ell}h_{j\ell}$ play a key role in determining the asymptotic properties of the ordered eigenvalues $\lambda_{(i)}$.

Theorem. If $\alpha \in (0,2)$, then

$$a_{np}^{-2} \max_{i=1,\dots,p} |\lambda_{(i)} - \delta_{(i)}| \xrightarrow{\mathbb{P}} 0, \quad n \to \infty,$$

where $\delta_{(1)} \ge \dots \ge \delta_{(p)}$ are the ordered values of the set $\{D_{(i)}v_{(j)} : i \le p; j \ge 1\}$.

Point Process Convergence

Let (E_i) be iid standard exponential random variables and $\Gamma_i = E_1 + \ldots + E_i$. Then we have the point process convergence

$$\sum_{i=1}^{p} \varepsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{d} \sum_{i=1}^{\infty}\sum_{j=1}^{r} \varepsilon_{\Gamma_i^{-2/\alpha} v_j}. \quad (4)$$

An application of (4) then yields for every fixed integer $k \ge 1$,

$$a_{np}^{-2}(\lambda_{(1)}, \dots, \lambda_{(k)}) \xrightarrow{d} (d_{(1)}, \dots, d_{(k)}),$$

where $d_{(1)} \ge \dots \ge d_{(k)}$ are the k largest ordered values of the set $\{\Gamma_i^{-2/\alpha} v_j : i, j \ge 1\}$. In particular we find

$$d_{(1)} = v_1\Gamma_1^{-2/\alpha} \text{ and } d_{(2)} = v_2\Gamma_1^{-2/\alpha} \vee v_1\Gamma_2^{-2/\alpha}. \quad (5)$$

Example

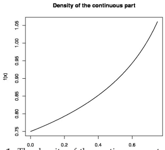

Figure 1: The density of the continuous part of Y defined in (2) with $\alpha = 1.5$.

Assume that $\alpha \in (0,2)$ and

$$X_{it} = Z_{it} + Z_{i,t-1} - 2(Z_{i-1,t} - Z_{i-1,t-1}). \quad (1)$$

The matrix M has rank 2 and the non-negative eigenvalues $v_1 = 8$ and $v_2 = 2$. The limit point process in (4) is

$$\sum_{i=1}^{\infty} \varepsilon_{8\Gamma_i^{-2/\alpha}} + \sum_{i=1}^{\infty} \varepsilon_{2\Gamma_i^{-2/\alpha}}.$$

By (5) we get

$$a_{np}^{-2}\lambda_{(2)} \xrightarrow{d} 2\Gamma_1^{-2/\alpha} \vee 8\Gamma_2^{-2/\alpha}.$$

Since Γ_1/Γ_2 has a standard uniform distribution, we can easily compute

$$\mathbb{P}(2\Gamma_1^{-2/\alpha} > 8\Gamma_2^{-2/\alpha}) = 2^{-\alpha} \in (1/4, 1).$$

The self-normalized spectral gap

$$\frac{\lambda_{(1)} - \lambda_{(2)}}{\lambda_{(1)}}$$

converges in distribution to a random variable which has the same distribution as

$$Y := 3/4 I_{\{U<2^{-\alpha}\}} + (1 - U^{2/\alpha}) I_{\{U>2^{-\alpha}\}}, \quad (2)$$

where U is standard uniformly distributed. Y has an atom at $3/4$ with point mass $2^{-\alpha}$. The ratio of the two largest eigenvalues is of special interest.
In the case of independent rows it was shown that $\lambda_{(2)}/\lambda_{(1)} \to U^{\alpha/2}$ in distribution. In our model, however, the rows are dependent and the limit takes the form

$$c^{\alpha/2} I_{\{U<c\}} + U^{\alpha/2} I_{\{U>c\}}$$

for a non-negative constant c. To confirm this limit structure we simulate the ratio $(\lambda_{(2)}/\lambda_{(1)})^{2/\alpha}$ from the model (1) for $\alpha = 1.5$. The theoretical limit variable is

$$(1 - Y)^{2/\alpha} = 0.35 I_{\{U<0.35\}} + U_{\{U>0.35\}}. \quad (3)$$

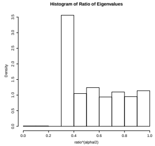

Figure 2: The histogram of $(\lambda_{(2)}/\lambda_{(1)})^{2/\alpha}$ based on 1000 replications from the model (1) with noise given by a t-distribution with $\alpha = 1.5$ degrees of freedom, $n = 1000$ and $p = 200$.

A histogram based on realizations of the true limit variable (3) would look very similar.

References, Future Research & Contact Information

1. Davis, Heiny, Mikosch, Xie (2016). *Extreme value analysis for the sample autocovariance matrices of heavy-tailed multivariate time series.* Extremes, to appear.
2. Heiny, Mikosch. *Eigenvalues and eigenvectors of heavy-tailed sample covariance matrices with general growth rates: the iid case.* Submitted.

- Autocovariance matrix.
- Eigenvectors.
- Other non-linear structures of X_{it}.
- Sample correlation matrices.

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Autocovariance matrices

Autocovariance function and singular values.
Let

$$\boldsymbol{X}_n(s) = (X_{i,t+s})_{i=1,\ldots,p,t=1,\ldots,n}\,, \quad n \geq 1\,,$$

then $\boldsymbol{X}_n = \boldsymbol{X}_n(0)$. The **autocovariance matrices** for lags $s \in \mathbb{N}_0$ are

$$\boldsymbol{X}_n(0)\boldsymbol{X}_n(s)'.$$

Limit theory for singular values of such matrices.

Sample Correlation Matrices

Assumptions: (X_{it}) iid, $p/n \to \gamma \in (0,1]$.
Define the $p \times p$ diagonal matrix $\mathbf{F} = (\mathrm{diag}(\boldsymbol{X}\boldsymbol{X}'))^{-1}$.
Sample correlation matrix $\mathbf{R}$:

$$\mathbf{R} = \mathbf{F}^{1/2}\boldsymbol{X}\boldsymbol{X}'\mathbf{F}^{1/2}$$

and its ordered eigenvalues

$$\mu_{(1)} \geq \cdots \geq \mu_{(p)} \,.$$

Note that the matrices $\mathbf{F}^{1/2}\boldsymbol{X}\boldsymbol{X}'\mathbf{F}^{1/2}$ and $\boldsymbol{X}\boldsymbol{X}'\mathbf{F}$ have the same eigenvalues.
The results on sample covariance matrices can be used to draw conclusions about the behavior of the eigenvalues of the sample correlation matrix.

Sample Correlation Matrices

By Weyl's inequality we have

$$\begin{aligned}\max_{i=1,\dots,p} |\mu_{(i)} - n^{-1}\lambda_{(i)}| &\le \|\boldsymbol{X}\boldsymbol{X}'\mathbf{F} - n^{-1}\boldsymbol{X}\boldsymbol{X}'\|_2 \\ &\le n^{-1}\|\boldsymbol{X}\boldsymbol{X}'\|_2\|n\mathbf{F} - \mathbf{I}\|_2 \\ &= n^{-1}\lambda_{(1)} \max_{i=1,\dots,p}\left|\frac{n}{\sum_{t=1}^n X_{it}^2} - 1\right|.\end{aligned} \tag{1}$$

If $\mathbb{E}[X^4] < \infty$,

$$\max_{i=1,\dots,p}\left|\frac{n}{\sum_{t=1}^n X_{it}^2} - 1\right| \xrightarrow{a.s.} 0.$$

This approach was used by [Jiang, 2004], and [Xiao and Zhou, 2010].

Sample Correlation Matrices under infinite fourth moment

Almost sure convergence of $\mu_{(1)}$ for symmetric X

If the iid entries X_{it} satisfy a moment condition which is "essentially"

$$n\mathbb{E}\Big[\frac{X_{11}^4}{D_1^2}\Big] \to 0\,,$$

then F_R converges to the Marčenko–Pastur law and

$$\lim_{n\to\infty} \mu_{(1)} = (1+\sqrt{\gamma})^2\,, \text{a.s.} \tag{2}$$

If

$$n\mathbb{E}\Big[\frac{X_{11}^4}{D_1^2}\Big] \not\to 0\,,$$

the empirical spectral distribution F_R does not converge to the Marčenko–Pastur law.

References I

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Materials of this talk

[1] Davis, R. A., Mikosch, T., Heiny, J., and Xie, X. (2016). Extreme value analysis for the sample autocovariance matrices of heavy-tailed multivariate time series. *Extremes.*

[2] Heiny, J. and Mikosch, T. (2016b). Eigenvalues and eigenvectors of heavy-tailed sample covariance matrices with general growth rates: the iid case. *Submitted.*

[3] Davis, R. A., Heiny, J., and Mikosch, T. (2016). Limit theory for the singular values of the sample autocovariance matrix function of multivariate time series. *In preparation.*

[4] Heiny, J. and Mikosch, T. (2016a). Almost sure convergence of the largest eigenvalue of the sample correlation matrix under infinite fourth moment. *In preparation.*