Stochastic Calculus w.r.t. Gaussian Processes

Conference on Ambit Fields and Related Topics

Aarhus, 15-18 August 2016.

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems
   - Multifractional Hull and White Model

Outline of the presentation

Multifractional Brownian Motion

Fractional Brownian motion (fBm)

A gaussian process more flexible than standard Brownian motion (A. Kolmogorov, 1949)

Definition

Let $H \in (0,1)$ be a real constant. A process $B^H := (B^H_t; t \in \mathbb{R}_+)$ is an fBm if it is centred, Gaussian, with covariance function given by:

$$\mathbb{E}[B^H_t B^H_s] = 1/2(t^{2H} + s^{2H} - |t-s|^{2H}). \; .$$

Properties

The process B^H verifies

- $B^H_0 = 0$, *a.s.*
- *For all* $t \geqslant s \geqslant 0$, $B^H_t - B^H_s$ *follows the law* $\mathcal{N}(0, (t-s)^{2H})$.
- *The trajectoiries de* B^H *are contiuous.*

Properties of fractional Brownian Motion

Properties

If B^H is a fBm, it verifies the following assertions:

- $X_t = \frac{1}{a^H} B^H_{at}$ *with* $a > 0$ *is un fBm (self-similarity);* $B^{1/2}$ *is a sBm.*
- *For* $H > 1/2$, $B^H_{t+h} - B^H_t$ *et* $B^H_{t+2h} - B^H_{t+h}$ *are positively corelated and* B^H *has a long terme dependance.*
- *For* $H < 1/2$, $B^H_{t+h} - B^H_t$ *et* $B^H_{t+2h} - B^H_{t+h}$ *are negatively corelated..*
- *A.s, in all point* t_0 *of* $\mathbb{R}_+$, *the regularity of* B^H *is constant and equal to* H.

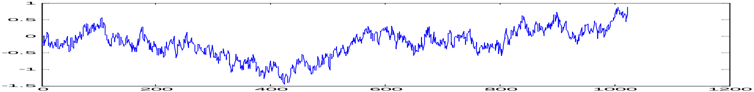
Because sBm and fBm are not differentiable, a good measure of their regularity, as a process, is the local Hölder exponent which is defined at every point t_0, by:

Definition

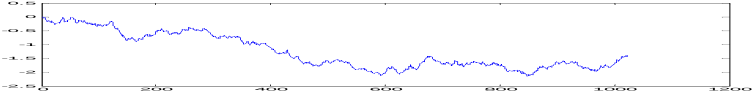
$$\alpha_{B^H}(t_0) := sup\{\alpha : \limsup_{\rho \to 0} \sup_{(s,t) \in B(t_0,\rho)^2} \frac{|B^H_t - B^H_s|}{|t-s|^\alpha} < +\infty\} = H.$$

Trajectories with different regularity (Fraclab)

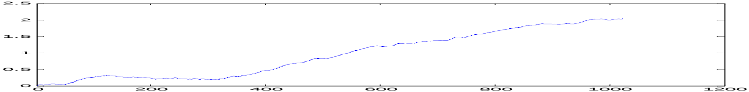
fBm with $H = 0.3$



fBm with $H = 0.5$



fBm with $H = 0.8$



Drawbacks of fBm

• long range dependence versus regularity of trajectories: model the increments present long range dependence only if $H > 1/2$.

• the regularity of trajectories remains the same along the time (equal to H)....

Multifractional Brownian Motion

What is Multifractional Brownian motion (mBm)?

A Gaussian process more flexible than the fBm: Lévy Véhel, Peltier (1995); Benassi, Jaffard and Roux (1997)

We here give the more recent definition of mBm given in[A].

Definition (Fractional Gaussian field)

A two parameters Gaussian process $(\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ *is said to be a fractional Gaussian field if, for every* $H \in (0,1)$, *the process* $(\mathbf{B}(t,H))_{t\in\mathbb{R}}$ *is a fractional Brownian motion.*

A multifractional Brownian motion is simply a "path" traced on a fractional Gaussian field.

Definition (Multifractional Brownian motion)

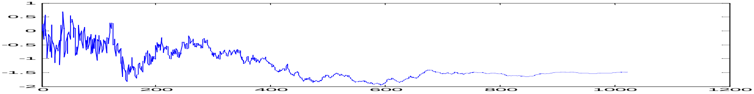
Let $h:\mathbb{R}\to(0,1)$ *be a deterministic measurable function and* $\mathbf{B} := (\mathbf{B}(t,H))_{(t,H)\in\mathbb{R}\times(0,1)}$ *be a fractional Gaussian field. The Gaussian process* $B^h := (\mathbf{B}(t,h(t)))_{t\in\mathbb{R}}$ *is called a mBm with functional parameter h.*

Example

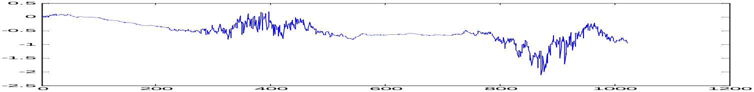
$$\mathbf{B}(t,H) := \frac{1}{c_{h(t)}} \int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{h(t)+1/2}} \widetilde{\mathbb{W}}(du).$$

Graphic Representations of mBm $\mathbf{B}$ for several functions h obtained thanks to the software Fraclab

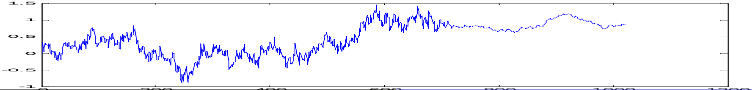
mBm with $h(t) := 0.1 + 0.8t$



mBm with $h(t) := 0.5 + 0.3\sin(4\pi t)$



mBm with $h(t) := 0.3 + 0.3(1 + \exp^{(-100(t-0.7))})^{-1}$



Gaussian process for which one wants to define a stochastic calculus

More generally, for every Gaussian process $G = (G_t)_{t \in \mathbb{R}_+}$ which can be written under the form:

$$G_t := \int_{\mathbb{R}} g_t(u)\ dB_u =< ., g_t >,$$

where $g_t \in L^2(\mathbb{R}_+)$.

How can one define a stochastic integral, and a stochastic calculus with respect to G?

Outline of the presentation

The stochastic calculus developed for continuous semi-martingales can not be applied for mBm

- Since fBm, mBm and more generally G, are not semimartingales, we can not use standard stochastic calculus for them.

$\Longrightarrow$ We then have to develop a different (new) stochastic calculus and hence new methods....

- How can one construct an integral with respect to fBm, mBm and G (especially when G is not a semimartingale)?

What do we want to do with an integral *w.r.t.* G?

To be able to solve S.D.Es that come from, e.g.

- finance
- physics
- Medicine
- Geology

Several approaches in order to obtain a stochastic calculus with respect to fractional Brownian motion (fBm)

Probabilistic approaches	**Deterministic approaches**
Malliavin Calculus Decreusefond, Üstunel, Alos, Mazet, Nualart...	Fractional Integration Zähle, Feyel & de la Pradelle
White Noise Theory Hida,Kuo,Elliott,Bender,Sulem...	Rough Path theory Coutin, Nourdin, Gubinelli
Enlargment of filtrations Jeulin, Yor	Extended integral via regularization Russo & Vallois

Approaches in order to obtain a stochastic calculus with respect to multifractional Brownian motion (mBm)

In the probabilistic approaches:

- The one provided by[B] using the divergence type integral (Malliavin Calculus), which is valid for Voltera processes.
- Only for mBm, (see[C]).

[B] **nualart**.
[C] **JLJLV1**; **EHJLJLV**.

Outline of the presentation

Background on White Noise Theory

$(\Omega, \mathcal{F}, \mu) = (\mathscr{S}'(\mathbb{R}), \mathcal{B}(\mathscr{S}'(\mathbb{R})), \mu)$, where μ is the unique probability measure such that for all $f \in L^2_{\mathbb{R}}(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, the map

$$
\begin{array}{rccc}
< ., f >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})} : & (\Omega, \mathscr{F}) & \to & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \\
& \omega & \mapsto & < \omega, f >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})}
\end{array}
$$

is a real r.v following the law $\mathcal{N}(0, \|f\|^2_{L^2_{\mathbb{R}}(\mathbb{R})})$. For every n in $\mathbb{N}$, define the n^{th} Hermite function by:

$$e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2});$$

$(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$

Thorem-defintion (Test functions and Hida distributions spaces)

There exist two topological spaces noted $(\mathcal{S})$ and $(\mathcal{S})^*$ such that we have

- $$(\mathcal{S}) \subset (L^2) := L^2(\Omega, \mathcal{F}, \mu) \subset (\mathcal{S})^*$$
- *$(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$ and we will note $<\,,\,>$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$.*
- *If Φ belongs to (L^2) then we have the equality* $\ll \Phi, \varphi \gg \; = < \Phi, \varphi >_{(L^2)} = \mathbb{E}[\Phi\ \varphi]$.

Remark

We call $(\mathcal{S})$ the test function space and $(\mathcal{S})^*$ the Hida distributions space.

Definition (Convergence in $(\mathcal{S})^*$)

For every $\Phi_n := \sum_{k=0}^{+\infty} a_k^{(n)} < ., e_k >$, one says that $(\Phi_n)_{n\in\mathbb{N}}$ converge to $\Phi := \sum_{k=0}^{+\infty} a_k < ., e_k >$,

- in $(\mathcal{S})^*$ if $\exists p_0$ s.t. $\lim_{n\to+\infty} \sum_{k=0}^{+\infty} \frac{(a_k - a_k^{(n)})^2}{(2k+2)^{2p_0}} = 0$.
- in $(\mathcal{S})$ if $\forall p_0$, $\lim_{n\to+\infty} \sum_{k=0}^{+\infty} \left(a_k - a_k^{(n)}\right)^2 (2k+2)^{2p_0} = 0$.

Definition (stochastic distribution process)

A measurable function $\Phi : I \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$–process, or a Hida process.

Definition (derivative in $(\mathcal{S})^*$)

Let $t_0 \in I$. A stochastic distribution process $\Phi : I \to (\mathcal{S})^*$ is said to be differentiable at t_0 if the quantity $\lim_{r\to 0} \; r^{-1} \, (\Phi(t_0 + r) \; - \; \Phi(t_0))$ exists in $(\mathcal{S})^*$. We note $\frac{d\Phi}{dt}(t_0)$ the $(\mathcal{S})^*$-derivative at t_0 of the stochastic distribution process Φ. Φ is said to be differentiable over I if it is differentiable at t_0 for every t_0 in I.

Definition (integral in $(\mathcal{S})^*$)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto < \Phi(u), \; \varphi >$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, dt)$. Then there exists an unique element in $(\mathcal{S})^*$, noted $\int_{\mathbb{R}} \Phi(u)du$ such that

$$\ll \int_{\mathbb{R}} \Phi(u)du, \varphi \gg = \int_{\mathbb{R}} \ll \Phi(u), \varphi \gg \; du \qquad \text{for all } \varphi \text{ in } \; (\mathcal{S}).$$

Assumptions $(\mathcal{A}_1)$ & $(\mathcal{A}_2)$

The Gaussian process $G := (< ., g_t >)_{t\in\mathbb{R}}$ being fixed, define

$$g : \begin{array}{ccc} \mathbb{R} & \to & \mathscr{S}'(\mathbb{R}) \\ t & \mapsto & g(t) := g_t \end{array}$$

In the sequel, we make the following assumption:

$(\mathcal{A}_1)$ The map g is differentiable on $\mathbb{R}$ (one notes $g'_t := g'(t)$).

$(\mathcal{A}_2)$ $\exists\ q \in \mathbb{N}^*$ *s.t.* the map $t \mapsto |g'_t|_{-q} \in L^1_{\text{loc}}(\mathbb{R})$,

where $|f|^2_{-q} := \sum_{k=0}^{+\infty} < f, e_k >^2 (2k+2)^{-2q}, \quad \forall (f,q) \in L^2(\mathbb{R}) \times \mathbb{N}$.

Remark

We will note and call Gaussian White Noise the process $(W_t^{(G)})_{t\in\mathbb{R}}$ defined by $W_t^{(G)} := < ., g'_t >$, where the equality holds in $(\mathcal{S})^*$. We will sometimes note $\frac{dG_t}{dt}$ instead of $W_t^{(G)}$.

Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are fulfilled for standard, fractional and multifractional Brownian motions. Indeed:

Example: Derivative of Brownian motion

Example (white noise)

The process defined by $B_t := < ., \mathbb{1}_{[0;t]} >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})}$ *is a Brownian motion and has the following expansion, in* $(\mathcal{S})^*$*:*

$$B_t = \sum_{k=0}^{+\infty} \Big(\int_0^t e_k(s)ds \Big) < ., e_k > .$$

It is natural to think to define the derivative of B *with respect to time, denoted* $(W_t)_{t\in[0,1]}$*, by setting:*

$$W_t := \sum_{k=0}^{+\infty} e_k(t) < ., e_k > .$$

The map $t \mapsto W_t$ *is the derivative, in sense of* $(\mathcal{S})^*$*, of the Bronwian motion. It is called white noise process and is sometimes denoted by* $\frac{dB}{dt}(t)$ *or* $\dot{B}_t$*.*

One can do the same for fBm & mBm

The operator M_H crucial to define both fBm and mBm

For all H in $(0,1)$, we define on the space $\mathcal{E}(\mathbb{R})$ of step functions the operator M_H by

$$\widehat{M_H(f)}(x) = x^{1/2-H}\,\widehat{f}(x), \text{ for } a.e.\ x \in \mathbb{R}.$$

The map $M_H : (\mathcal{E}(\mathbb{R}), <,>_H) \to (L^2_{\mathbb{R}}(\mathbb{R}), <,>_{L^2_{\mathbb{R}}(\mathbb{R})})$ is an isometry and can then be extended from $\mathcal{E}(\mathbb{R})$ to

$$L^2_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) \mid \widehat{u} \in L^1_{loc}(\mathbb{R}) \text{ and s.t } ||u||^2{}_{L^2_H(\mathbb{R})} < +\infty\},$$

where $\|u\|^2{}_H := \|u\|^2{}_{L^2_H(\mathbb{R})} := \beta_H^2 \int_{\mathbb{R}} |u|^{1-2H} |\widehat{f}(u)|^2 du.$

Example of fractional white noise

Example (fractional white noise)

Let $H \in (0,1)$, the process defined by $B^H_t := < ., M_H(\mathbb{1}_{[0;t]}) >_{\mathscr{S}'(\mathbb{R}), L^2_{\mathbb{R}}(\mathbb{R})}$ is a fBm and has the following expansion in $(\mathcal{S})^*$:

$$B^H_t = \sum_{k=0}^{+\infty} \left(\int_0^t M_H(e_k)(s)ds \right) < ., e_k > .$$

We define the fractional white noise W^H by

$$W^H_t := \sum_{k=0}^{+\infty} M_H(e_k)(t) < ., e_k > .$$

The map $t \mapsto W^H_t$ is the derivative, in sense of $(\mathcal{S})^*$, of the process B^H. We call it multifractional White Noise process and denote it sometimes by $\frac{dB^H}{dt}(t)$.

More generally, Gaussian white noise

Gaussian white Noise

Let $G := (< ., g_t >)_{t\in\mathbb{R}}$ be a Gaussian process that fulfills assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, G has the following expansion in $(\mathcal{S})^*$:

$$G_t = \sum_{k=0}^{+\infty} \Big(< g_t, e_k >_{L^2(\mathbb{R})} \Big) < ., e_k > .$$

We define the Gaussian white noise W^G by

$$W_t^{(G)} := \sum_{k=0}^{+\infty} \Big(< g_t', e_k >_{L^2(\mathbb{R})} \Big) < ., e_k > .$$

The map $t \mapsto W_t^{(G)}$ is the Hida derivative, of the process G. We call it Gaussian White Noise process and denote it sometimes by $\frac{dG}{dt}(t)$.

Outline of the presentation

Recall

Definition (integral in $(\mathcal{S})^*$)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ *is weakly in* $L^1(\mathbb{R}, dt)$, *i.e assume that for all* φ *in* $(\mathcal{S})$, *the mapping* $u \mapsto \ll \Phi(u),\ \varphi \gg$ *from* $\mathbb{R}$ *to* $\mathbb{R}$ *belongs to* $L^1(\mathbb{R}, dt)$. *Then there exists an unique element in* $(\mathcal{S})^*$, *noted* $\int_{\mathbb{R}} \Phi(u)du$ *such that*

$$\ll \int_{\mathbb{R}} \Phi(u)du, \varphi \gg = \int_{\mathbb{R}} \ll \Phi(u), \varphi \gg \ du \quad \text{for all } \varphi \text{ in } (\mathcal{S}).$$

Stochastic integral with respect to G

Definition (Wick-Itô integral w.r.t. Gaussian process)

Let $X : \mathbb{R} \to (\mathcal{S})^*$ be a process s.t. the process $t \mapsto X_t \diamond W_t^{(G)}$ is $(\mathcal{S})^*$-integrable on $\mathbb{R}$. The process X is then said to be dG-integrable on $\mathbb{R}$ or integrable on $\mathbb{R}$, with respect to the Gaussian process G. The integral on $\mathbb{R}$ of X with respect to G is defined by:

$$\int_{\mathbb{R}} X_s \, d^{\diamond} G_s := \int_{\mathbb{R}} X_s \diamond W_s^{(G)} \, ds.$$

For any Borel set I of $\mathbb{R}$, define $\int_I X_s \, d^{\diamond} G_s := \int_{\mathbb{R}} \mathbb{1}_I(s) \, X_s \, dG_s$.

Properties

- *The Wick-Itô integral of an $(\mathcal{S}^*)$-valued process, with respect to G is then an element of $(\mathcal{S})^*$.*
- *Let (a,b) in $\mathbb{R}^2$, $a < b$. Then $\int_a^b \, d^{\diamond} G_u = G_b - G_a$ almost surely.*
- *The Wick-Itô integration with respect to G is linear.*
- *Let $X : I \to (\mathcal{S}^*)$ be a dG-integrable process over I & assume $\int_I X(s) \, d^{\diamond} G_s$ belongs to (L^2). Then*

$$\mathbb{E}\left[\int X(s) \, d^{\diamond} W_s^G\right] = 0.$$

Example I: A simple comptutation

Computation of $\int_0^T G_t \; dG_t$

Let $T > 0$and assume that $t \mapsto R_{t,t} := \mathrm{E}[G_t^2]$ is upper-bounded on $[0, T]$, then the following equality holds almost surely and in (L^2).

$$\int_0^T G_t \; d^\diamond G_t = \tfrac{1}{2} \left(G_T^2 - R_{T,T} \right)$$

Proof:

Existence of both sides, using S-transfrom.

$$S(\int_0^T G_t \; d^\diamond G_t)(\eta) = \int_0^T S(G_t)(\eta) \; S(W_t^{(G)})(\eta) \; dt = \int_0^T < g_t, \eta >< g_t', \eta > \, dt$$

$$= \tfrac{1}{2}(S(G_T)(\eta))^2 = S(\tfrac{1}{2} G_T^{\diamond 2})(\eta),$$

Wick product has replaced ordinary product.

Example II: A simple SDE, The Gaussian Wick exponential

The Gaussian Wick exponential

Let us consider the following Gaussian stochastic differential equation

$$(\mathcal{E}) \begin{cases} dX_t = \alpha(t) X_t \ dt + \beta(t) X_t \ d^{\diamond} G_t \\ X_0 \in (\mathcal{S})^*, \end{cases}$$

where t belongs to $\mathbb{R}_+$ and where $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ are two deterministic continuous functions. This equation is a shorthand notation for $X_t = X_0 + \int_0^t \ \alpha(s) \ X_s \ ds + \int_0^t \ \beta(s) \ X_s \ d^{\diamond} G_s$, where the equality holds in $(\mathcal{S})^*$.

Theorem

The process $Z := (Z_t)_{t \in \mathbb{R}}$ defined by

$$Z_t := X_0 \diamond \exp^{\diamond} \left(\int_0^t \alpha(s) \ ds + \int_0^t \beta(s) \ d^{\diamond} G_s \right), \qquad t \in \mathbb{R}_+, \tag{2.1}$$

is the unique solution, in $(\mathcal{S})^*$, of $(\mathcal{E})$.

Outline of the presentation

Comparison with Malliavin calculus or divergence integral I

Comparison

Our goal is now to compare the Wick-Itô integral with respect to G we just define to the divergence integral with respect to G, defined and studied in[a], on a compact set.

[a] **nualart**; **nu05**.

We will denote $\int_0^T u_s \, \delta G_s$ the divergence integral with respect to G defined in[D]

[D] **nualart**; **nu05**.

Comparison with Malliavin calculus or divergence integral II

Theorem: Comparison between Wick-Itô & divergence integral

Let u be a process in $L^2(\Omega, L^2([0,T]))$. If u belongs to the domain of the divergence of G, then u is Wick-Itô integrable on $[0,T]$ with respect to G. Moreover one has the equality

$$\int_0^T u_s \, \delta G_s = \int_0^T u_s \; d^{\diamond} G_s.$$

Comparison with Malliavin calculus or divergence integral III

Remark

Note that $\int_0^T u_s\ \delta G_s = \int_0^T u_s\ dG_s$ is not true in general.
For example, $\int_0^T B_t^H\ dB_t^H$ exist and is equal to $\frac{1}{2}((B_T^H)^2 - T^{2H})$ for every $H \in (0,1)$ but $\int_0^T\ B_t^H \delta B_t^H$ does not even exist when $H < 1/4$.

Finally, the only thing one can say, in general, is that we have the dense inclusion $L^2(\Omega, L^2([0,T])) \cap \mathbb{D}\text{om}(\delta_G) \subset \Lambda$, where

- $\mathcal{H}_T := \overline{\text{span}\{\mathbb{1}_{[0,t]},\ t \in [0,T]\}}^{(L^2)}$
- $\Lambda := \{u \in L^2(\Omega; \mathcal{H}_T);\ u$ is Wick-Itô integrable w.r.t. G & s.t. $\int_0^T u_s\ \ d^{\diamond} G_s \in L^2(\Omega)\}$.

- $\mathbb{D}\text{om}(\delta_G) := \{u \in L^2(\Omega; \mathcal{H}_T);$ & s.t. $\int_0^T u_s\ \delta G_s$ *exists* and $\in L^2(\Omega)\}$.

Conclusion on the comparison

Remark (Comparison with Itô integral)

- *When G is a Brownian motion (or even a Gaussian martingale), the Wick-Itô integral with respect to G is nothing but the classical Itô integral, provided X is Itô-integrable (which implies in particular that X is a previsible process).*

Remark

- *When G is a fractional (resp. multifractional) Brownian motion, the Wick-Itô integral with respect to G coincide with the fractional (resp. multifractional) Wick-Itô integral defined in[a] (resp. in[b]).*
- *The Wick-Itô integral fully generalize the one provided by Viens-Mocioalca (2005).*

---

[a] **ell**; **bosw**; **ben1**; **ben2**.
[b] **JLJLV1**; **EHJLJLV**; **JL13**.

Outline of the presentation

An Itô formula in (L^2)

Denote $t \mapsto R_t$ the variance function of G.

Theorem: Itô formula in (L^2)

Let $T > 0$ and f be a $C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ function. Furthermore, assume that $t \mapsto R_{t,t}$ is differentiable and that there are constants $C \geqslant 0$ and $\lambda < (4 \max_{t \in [0,T]} R_t)^{-1}$ such that for all (t,x) in $[0,T] \times \mathbb{R}$,

$$\max_{t \in [0,T]} \left\{ |f(t,x)|, \left|\frac{\partial f}{\partial t}(t,x)\right|, \left|\frac{\partial f}{\partial x}(t,x)\right|, \left|\frac{\partial^2 f}{\partial x^2}(t,x)\right| \right\} \leqslant C e^{\lambda x^2}.$$

Assume moreover that the map $t \mapsto R_t$ is both continuous and of bounded variations on $[0,T]$. Then, for all t in $[0,T]$, the following equality holds in (L^2):

$$f(T, G_T) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, G_t)\, dt + \int_0^T \frac{\partial f}{\partial x}(t, G_t)\, d^{\diamond} G_t + \frac{1}{2} \int_0^T \frac{\partial^2 f}{\partial x^2}(t, G_t)\, dR_t.$$

Remark

It is clear that one can extend the definition of integral in $(\mathcal{S})^*$ to the case where the measure considered is not the Lebesgue measure but a difference of two positive measures, denoted m. The definition of integral in $(\mathcal{S})^*$ wrt m is then:

Definition (integral in $(\mathcal{S})^*$ *wrt* m)

Assume that $\Phi : \mathbb{R} \to (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, m)$, i.e assume that for all φ in $(\mathcal{S})$, the mapping $u \mapsto <\Phi(u),\ \varphi>$ from $\mathbb{R}$ to $\mathbb{R}$ belongs to $L^1(\mathbb{R}, |m|)$. Then there exists an unique element in $(\mathcal{S})^*$, denoted $\int_{\mathbb{R}} \Phi(u)\ m(du)$ such that

$$< \int_{\mathbb{R}} \Phi(u)\ m(du), \varphi > = \int_{\mathbb{R}} < \Phi(u), \varphi > \ m(du) \qquad \text{for all } \varphi \text{ in } (\mathcal{S}).$$

Theorem: Tanaka formula for G

Let $T > 0$ be such that $[0, T] \subset$ and c be real number. Assume that the map $t \mapsto R_t$ is both continuous and of bounded variations on $[0, T]$ and such that:

(i) $t \mapsto R_t^{-1/2} \in L^1([0, T], dR_t)$,

(ii) $\exists\ q \in$ such that $t \mapsto |g_t'|_{-q}\ R_t^{-1/2} \in L^1([0, T], dt)$,

(iii) $\lambda(\mathcal{Z}_R^T) = \alpha_R(\mathcal{Z}_R^T) = 0$.

Then, the following equality holds in (L^2):

$$|G_t - c| = |c| + \int_0^T \operatorname{sign}(G_t - c)\ G_t + \int_0^T \delta_{\{c\}}(G_t)\ dR_t, \tag{3.1}$$

where the function sign is defined on $\mathbb{R}$ by $\operatorname{sign}(x) := \mathbb{1}_{\mathbb{R}_+^*}(x) - \mathbb{1}_{\mathbb{R}_-}(x)$ and where $\delta_{\{a\}}(G_t)$ is the stochastic distribution defined by:

$$\delta_{\{c\}}(G_t) := \frac{1}{\sqrt{2\pi R_t}} \sum_{k=0}^{+\infty} \frac{1}{k!\ R_t^k} < \delta_{\{c\}}, \xi_{t,k} > I_k\left(g_t{}^{\otimes k}\right),$$

$$\text{with}\quad \xi_{t,k}(x) := \pi^{1/4}(k!)^{1/2} R_t^{k/2} \exp\{-\tfrac{x^2}{4R_t}\} e_k(x/(\sqrt{2R_t})),$$

About Itô and Tanaka formula

The results given in the last thee slides generalize the results provided

- for fBm by C. Bender[E]
- for mBm by J.L & J.Lévy Véhel [F]

[E] **ben1**; **ben2**.
[F] **JLJLV1**.

Outline of the presentation

Definition (non weighted local time of G)

The (non weighted) local time of G at a point $a \in \mathbb{R}$, up to time $T > 0$, denoted by $\ell_T^{(G)}(a)$, is defined by:

$$\ell_T^{(G)}(a) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \lambda(\{s \in [0, T];\ G_s \in (a-\varepsilon, a+\varepsilon)\}),$$

where λ denotes the Lebesgue measure on $\mathbb{R}$ and where the limit holds in $(\mathcal{S})^*$, when it exists.

Proposition

Let $T > 0$. Assume that the variance map $s \mapsto R_s$ is continuous on $[0, T]$ and s.t. $\lambda(\mathcal{Z}_R^T) = 0$, then:

1. *The map $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ for every $a \in \mathbb{R}^*$. The map $s \mapsto \delta_0(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ if $s \mapsto R_s^{-1/2}$ belongs to $L^1([0, T])$.*
2. *The following equality holds in $(\mathcal{S})^*$:*

$$\ell_T^{(G)}(a) = \int_0^T \delta_a(G_s)\ ds, \tag{3.2}$$

for every $a \in \mathbb{R}^*$; and, also for $a = 0$ if $s \mapsto R_s^{-1/2} \in L^1([0, T])$.

Thorem-defintion (weighted local time of G)

Let $T > 0$. Assume that the map $s \mapsto R_s$ is continuous and of bounded variations on $[0, T]$ and such that $\alpha_R()$ is equal to 0. Then:

1. *The map $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$- integrable on $[0, T]$ with respect to the measure α_R, for every $a \in \mathbb{R}^*$. The map $s \mapsto \delta_0(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ with respect to the measure α_R, if $s \mapsto R_s^{-1/2}$ belongs to $L^1([0, T], dR_s)$.*

2. *For every $a \in \mathbb{R}$, when $s \mapsto \delta_a(G_s)$ is $(\mathcal{S})^*$-integrable on $[0, T]$ with respect to the measure α_R, one can define the weighted local time of G at point a, up to time T, denoted $\mathscr{L}_T^{(G)}(a)$, as being the $(\mathcal{S})^*$- process defined by setting:*

$$\mathscr{L}_T^{(G)}(a) := \int_0^T \delta_a(G_s)\, dR_s,$$

where the equality holds in $(\mathcal{S})^*$.

Weighted and non-weighted local times are (L^2) random variables

Denote $\mathscr{M}_b(\mathbb{R})$ the set of positive Borel functions defined on $\mathbb{R}$.

Theorem: (Occupation time formula)

Let $T > 0$ and assume the variance map $s \mapsto R_s$ is continuous on $[0, T]$.

(i) Assume that $s \mapsto R_s^{-1/2} \in L^1([0,T])$ and that $\lambda_R(\mathcal{Z}_R^T) = 0$. If $\mathrm{E}\big[\int_{\mathbb{R}} \big|\int_0^T e^{i\xi G_s}\, ds\big|^2\, d\xi\big] < +\infty$, then the map $a \mapsto \ell_T^{(G)}(a)$ belongs to $L^2(\lambda \otimes \mu)$, where λ denotes the Lebesgue measure. Moreover one has the following equality, valid for μ-a.e. ω in Ω,

$$\forall\ \Phi\ \in \mathscr{M}_b(\mathbb{R}),\ \int_0^T \Phi(G_s(\omega))\, ds = \int_{\mathbb{R}} \ell_T^{(G)}(y)(\omega)\ \Phi(y)\, dy.$$

(ii) Assume that $s \mapsto R_s$ is of bounded variation on $[0,T]$, and such that $s \mapsto R_s^{-1/2} \in L^1([0,T], dR_t)$. Assume moreover that $\alpha_R(\mathcal{Z}_R^T) = 0$. If $\mathrm{E}\big[\int_{\mathbb{R}} \big|\int_0^T e^{i\xi G_s}\, dR_s\big|^2\, d\xi\big] < +\infty$, then $a \mapsto \mathscr{L}_T^{(G)}(a)$ belongs to $L^2(\lambda \otimes \mu)$. Moreover one has the following equality, valid for μ-a.e. ω in Ω,

$$\forall\ \Phi \in \mathscr{M}_b(\mathbb{R}),\ \int_0^T \Phi(G_s(\omega))\, dR_s = \int_{\mathbb{R}} \mathscr{L}_T^{(G)}(y)(\omega)\ \Phi(y)\, dy.$$

The previous results are, in particular valid, when one consider fBm or mBm.

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems
   - Multifractional Hull and White Model

An open problem I

Consider the fractional Brownian motion B^H its non weighted and weighted local times: $\ell^H := (\ell^H_t)_{t\in[0,T]}$ and $\mathscr{L}^H := (\mathscr{L}^H_t)_{t\in[0,T]}$ defined by:

$$\ell^H_t := \int_0^T \delta_{\{a\}}(B^H_t)\ dt \quad \& \quad \mathscr{L}^H_t := \int_0^T \tfrac{d}{dt}[t^{2H}]\ \delta_{\{a\}}(B^H_t)\ dt$$

Question

What can one say about the behavior of process ℓ^H?

More precisely, following the Result of Yor 1983 [G]

Theorem Yor - 1983

$$\Big\{\beta_t, \ell_t(a), \tfrac{\lambda^{1/2}}{2}(\ell_t(a/\lambda) - \ell_t(0))\,;(t,a)\in\mathbb{R}^2_+\Big\} \xrightarrow[\lambda\to\infty]{\text{in law}} \Big\{\beta_t, \ell_t(a), \mathbf{B}(\ell_t(0), a)\,;(t,a)\in\mathbb{R}^2_+\Big\}$$

where $\beta = (\beta_t)_{t\in[0,T]}$ is a Brownian motion and $\ell = (\ell_t(a))_{t\in[0,T]}$ is the local time of B at point a until time t and $\mathbf{B}$ a Brownian sheet, independent of the Brownian motion β.

[G] **Yor83**.

An open problem II

How can we translate this result in the world of fractional Brownian motion?

Conjecture

$$\left\{\beta^H_t, \ell^H_t(a), \tfrac{\lambda^s}{2}(\ell^H_t(a/\lambda) - \ell^H_t(0)), \ ; \ (t,a) \in \mathbb{R}^2_+\right\} \underset{\lambda\to\infty}{\overset{\text{in law}}{\longrightarrow}} \left\{\beta^H_t, \ell^H_t(a), \mathbf{B}(\ell^H_t(0), a), \ ; \ (t,a) \in \mathbb{R}^2_+\right\}$$

where $s := \frac{1}{2}(\frac{1}{H} - 1)$, β^H is a fractional Brownian motion and $\ell^H = (\ell^H_t(a))_{t\in[0,T]}$ is the local time of B^H at point a until time t and **B** a Brownian sheet, independent of the fractional Brownian motion β^H.

An open problem III

In general, what can one say about the convergence in law of

$$\left\{ G_t, \ell_t^{(G)}(a), \tfrac{\lambda^s}{2}(\ell_t^{(G)}(a/\lambda) - \ell_t^{(G)}(0)) \,; (t,a) \in \mathbb{R}_+^2 \right\},$$

for a Gaussian process $G := (< ., g_t >)_{t\in\mathbb{R}}$ that fulfills assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$?

Thank you for your attention!

Long-range dependence

Definition

A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if the autocovariance functions $\rho(n) := cov(X_k, X_{k+n})$ satisfy

$$\lim_{n\to\infty} c\ \rho(n)\ n^{\alpha} = 1$$

for some constant c and $\alpha \in (0,1)$. In this case, the dependence between X_k and X_{k+n} decays slowly as n tends to $+\infty$ and $\sum_{n=1}^{+\infty} \rho(n) = +\infty$.

The increments $X_k := B^H(k) - B^H(k-1)$ and $X_{k+n} := B^H(k+n) - B^H(k+n-1)$ of B^H have the long range dependence property for $H > 1/2$. Moreover, one has:

1. For $H > 1/2$, $\sum_{n=1}^{+\infty} \rho_H(n) = +\infty$
2. For $H < 1/2$, $\sum_{n=1}^{+\infty} |\rho_H(n)| < +\infty$

Applications to finance:

Multifractional stochastic volatility model

The problem of the volatility

Volatility in financial markets is both of crucial importance and hard to model in an accurate way. Black and Scholes model is not consistent with empirical findings.

- There is no reason to expect that instantaneous volatility should be constant or deterministic.

Popular models allowing for a varying volatility include ARCH models and their generalizations, stochastic volatility models and local volatility models.

- Local volatility models, in particular, enable the possibility to mimic in an exact way implied volatility surfaces.

The inconvenients of the present volatility models

Unfortunately such models do not take into account the fact that, while stocks do not typically exhibit correlations, volatility does display long-range correlations (see F.Comte and E.Renault).

Idea: Use the fractional stochastic volatility model given by

The Fractional Volatility Model

$$\begin{cases} dS_t = \mu(t, S_t)dt + S_t\sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right)dt + \gamma dB_t^H, \quad \sigma_0 > 0, \end{cases}$$

where W is a Brownian motion and B^H is an independent fBm under the historical probability. Such a model is

- consistent with the slow decay in the correlations of volatility observed in practice.

takes into account the fact that

- the implied volatility process is less persistent in the short term than a standard diffusion, while it is more persistent in the long run.

Nevertheless:

- the evolution in time of the smile is governed by the single constant parameter H.
- there is no reason to believe that the regularity of the volatility should be constant.
- one can not have, in the same time, an irregular volatility process (*i.e.* $H < 1/2$) and a long range dependance (*i.e.* $H > 1/2$).
- How can we have a hunch that multifractionality may be justified?

The Multifractional Volatility Model

It is given, in the risk-neutral setting by,

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right) dt + \gamma_h d^{\diamond} B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \\ d\langle W, W^\sigma \rangle_t = \rho dt, \qquad \text{where } \mathrm{B}^h \perp\!\!\!\perp W. \end{cases}$$

This model yields shapes of the smile at maturity T that are governed by a weighted average of the values of the function h up to time T.

Outline of the presentation

The multifractional Hull and White Model

We assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\,(\mu - \ln(\sigma_t))\,dt + \gamma_h d^{\diamond} B^h_t + \gamma_\sigma dW^\sigma_t, \quad \sigma_0 > 0, \end{cases} \tag{3.3}$$

where $\theta \geqslant 0$ and where W and W^σ are two standard Brownian motions and B^h is a mBm independent of W and W^σ.

The unique solution of (3.3) reads

$$\begin{cases} F_t = F_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right) \\ \sigma_s = \exp\left(\ln(\sigma_0)e^{-\theta s} + \mu\left(1 - e^{-\theta s}\right) + \gamma_\sigma \int_0^s e^{\theta(u-s)} dW^\sigma_u + \gamma_h \int_0^s e^{\theta(u-s)} d^{\diamond} B^h_u\right). \end{cases}$$

Forward start call option

We now consider the problem of pricing a forward start call option which payoff is $\left(\frac{F_T}{F_\tau} - K\right)_+$ for some fixed maturity $\tau \in [0, T]$.

We need to compute the risk-neutral expectation $\mathrm{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+\right]$.

$$\begin{aligned} \mathrm{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+\right] &= \mathrm{E}\left[\mathrm{E}\left[\left(\frac{F_T}{F_\tau} - K\right)_+ \middle| \mathcal{F}_T^{\sigma,h}\right]\right] \\ &= \mathrm{E}\left[\mathrm{PrimeBS}\left(F_{\tau,T}, \left((1-\rho^2)\frac{1}{T-\tau}\int_\tau^T \sigma_s^2 ds\right)^{\frac{1}{2}}, T-\tau, K\right)\right], \end{aligned}$$

where $F_{\tau,T} := \exp\left(\rho \int_\tau^T \sigma_s dW_s^\sigma - \frac{\rho^2}{2}\int_\tau^T \sigma_s^2 ds\right)$ and PrimeBS is the closed-form expression for the price of a Call option in the Black & Scholes model.

Functional Quantization

Since we are not able to compute exactly the quantity

$$\mathrm{E}\Big[\mathrm{PrimeBS}\Big(F_{\tau,T},\Big((1-\rho^2)\frac{1}{T-\tau}\int_\tau^T \sigma_s^2 ds\Big)^{\frac{1}{2}}, T-\tau, K\Big)\Big],$$

we have to approximate it and for doing it, to use

FUNCTIONAL QUANTIZATION

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems
   - Multifractional Hull and White Model

Functional Quantization

Definition

The quantization of a random variable X, valued in a reflexive separable Banach space $(E, |\cdot|)$, consists in its approximation by a random variable Y that is measurable with respect to X and that takes finitely many values in E (N let say). The minimization of the error reduces to the following optimization problem:

$$\min \left\{ \| \, |X - Y| \, \|_p, \; Y : \Omega \to E \text{ measurable with respect to } X, \; \text{card}(Y(\Omega)) \leqslant N \right\}.$$

This problem is solved by using projections. Moreover, if we denote by $\mathcal{E}_{N,p}(X, |\cdot|)$ the minimal L^p quantization error for the random variable X and the norm $|\cdot|$, we get:

$$\mathcal{E}_{N,p}(X, |\cdot|) = \min \left\{ \| |X - Y| \|_p, \; Y \text{ measurable with respect to } X \text{ and } \text{card}(Y(\Omega)) \leqslant N \right\}$$

Rate of decay of the quantization error for mBm

Theorem (L^2-mean regularity of mBm)

Let h be C^1 and B^h be an mBm with functional parameter h. Then there exists a positive constant M such that

$$\forall (s,t) \in [0,T]^2, \quad \mathrm{E}\left[\left(B^h_t - B^h_s\right)^2\right] \leqslant M\,|t-s|^{\,2 \inf\limits_{u\in[0,T]} h(u)},$$

H.Luschgy and G.Pagès criterion $\Rightarrow$

Corollary (Upper bound on the quantization error for mBm)

The upper bound on the quantization error for the mBm is given by

$$\mathcal{E}_{N,r}\left(B^h, |\cdot|_{L^p([0,T])}\right) = O\left(\log(N)^{-\inf\limits_{u\in[0,T]} h(u)}\right),$$

for every (r,p) in $(\mathbb{R}_+)^2$

One finally obtains the approximation:

$$\mathrm{E}\left[\left(\frac{F_T}{F_\tau}-K\right)_+\right] \approx \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} p_i^h p_j^\sigma \,\mathrm{PrimeBS}\left(F^{i,j}_{\tau,T},\left(\frac{(1-\rho^2)}{T-\tau}\int_\tau^T\left(\sigma^{i,j}(s)\right)^2 ds\right)^{\frac{1}{2}}, T-\tau, K\right),$$

where

- $F^{i,j}_{\tau,T} = \exp\left(\rho\int_\tau^T \sigma^{i,j}(s)d\chi_j^\sigma(s) - \rho\gamma^\sigma\frac{1}{2}\int_\tau^T\sigma^{i,j}(s)ds - \frac{\rho^2}{2}\int_\tau^T\left(\sigma^{i,j}(s)\right)^2 ds\right)$,
- $\sigma^{i,j}(t) := \exp\left(\ln(\sigma_0)e^{-\theta t} + \mu(1-e^{-\theta t}) + \gamma_\sigma e^{-\theta t} I_t^{e^{\theta\cdot}}(\chi_j^\sigma) + \gamma_h e^{-\theta t} I_t^{e^{\theta\cdot}}(\chi_j^h)\right)$,
- $I_t^f(g) := f(t)g(t) - \int_0^t f'(s)g(s)\,ds$.

and where $\left(\ln(\sigma^{i,j})\right)_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}$ and $(p_i^h p_j^\sigma)_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}$ are the paths and weights of a stationary quantizer of the process $\ln(\sigma)$.

- One controls its quadratic quantization error with the quantization error of $\widehat{W}^\sigma$ and $\widehat{B}^h$.

Outline of the presentation

1. Two particular Gaussian processes, Fractional and multifractional Brownian motion
   - Fractional and multifractional Brownian motions
   - Non semimartingales versus integration
2. Stochastic integral w.r.t. G in the White Noise Theory sense
   - Background on White Noise Theory
   - Stochastic Integral with respect to G
   - Comparison with Malliavin calculus or divergence integral
3. Miscellaneous formulas & some open problems
   - Miscellaneous formulas
     - Itô forumulas
     - Tanaka formula
   - Weighted and non weighted local times of G
   - Some open problems
   - Multifractional Hull and White Model

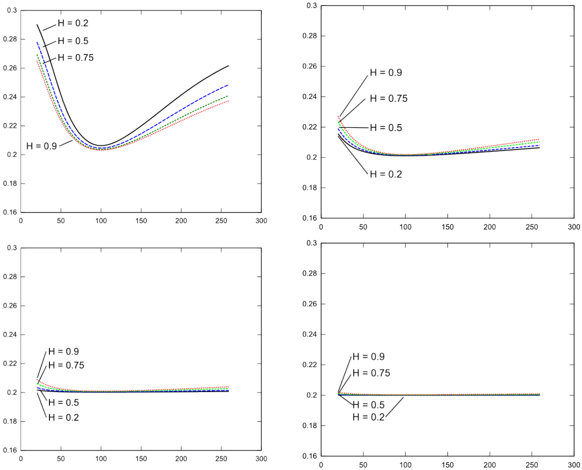


Figure: Comparisons of vanilla option volatility smiles for fBm with H in $\{0.2; 0.5; 0.7; 0.9\}$ for T in $\{1; 2.5; 5; 10\}$.

When h is constant:

for the short maturity $T = 1$ year, the smiles are more pronounced for small H and decrease as H increase, while the reverse is true for all maturities larger than one year.

- Thus, stronger correlations in the driving noise do translate in this model into a slower decrease of the smile as maturities increase, as noted in Comte-Renault.

However, with such an fBm-based model, an H larger than $1/2$ is needed to ensure long-range dependence and thus a more realistic evolution of the smile as compared to the Brownian case.

- This is not compatible with empirical graphs of the volatility which show a very irregular behaviour, and would require a small H.
- The local regularity of the volatility evolves in time, calling for a varying H, *i.e.* an mBm.

In a modelling with fBm, does not allow to control independently the shape of the smiles at different maturities.
This is possible with mBm $\Rightarrow$

We present results on the multifractional Hull & White model. We have computed the price as a function of strike for different maturities: 1, 2.5, 5 and 10 years. Driving noises were chosen in the class of fBms and mBms.

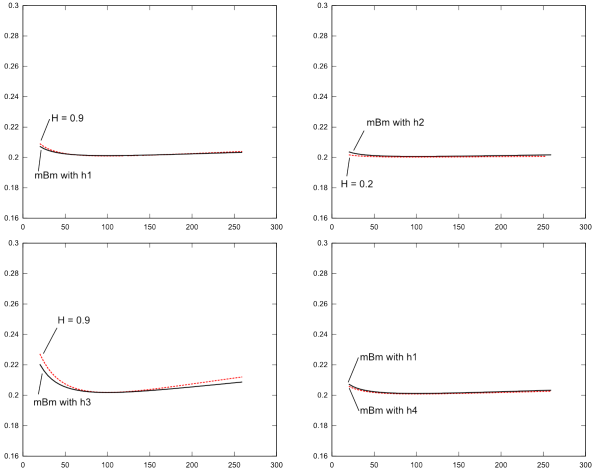


Figure: Comparisons of vanilla option volatility smiles for various fBm and mBm at several maturities. Top left: $h_1(5) = 0.9$ and $H = 0.9$. Top right: $h_2(5) = 0.2$ and $H = 0.2$. Bottom left: $h_3(2.5) = 0.9$ and $H = 0.9$. Bottom right: mBm with function

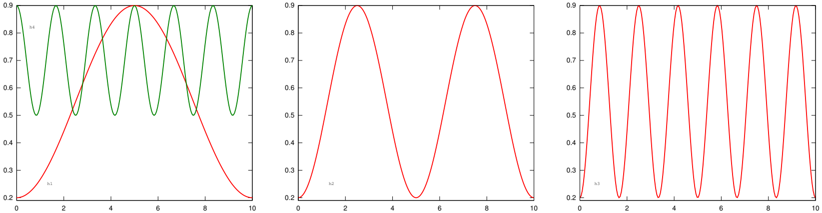


Figure: left: functions h_1 & h_4 ; middle: function h_2; right: function h_3.

1. An mBm with $h = h_1 = 0.35 \sin\left(\frac{2\pi}{10}\left(t + \frac{15}{2}\right)\right) + 0.55$.
2. An mBm with $h = h_2 = 0.35 \sin\left(\frac{2\pi}{5}\left(t + \frac{15}{4}\right)\right) + 0.55$.
3. An mBm with $h = h_3 = 0.35 \sin\left(\frac{6\pi}{5}\left(t + \frac{5}{4}\right)\right) + 0.55$.
4. An mBm with $h = h_4 = -0.2 \sin\left(\frac{6\pi}{5}\left(t + \frac{5}{4}\right)\right) + 0.7$.

And, finally, an mBm with $h = \widehat{h_{VolSP}}$, which corresponds to the regularity estimated on the S&P 500.

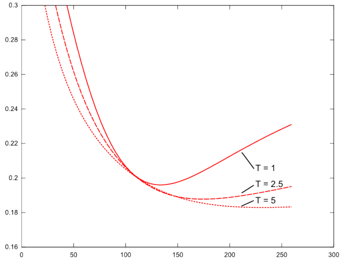


Figure: Vanilla option volatility smiles in the multifractional Hull & White model,

- (left): $\gamma_h = 0$, $\gamma_\sigma = 0.3$, $\theta = 0.3$, $\mu = \ln(0.2)$, $\sigma_0 = 0.2$, $F_0 = 100$, and $h = \widehat{h_{VolSP}}$ for maturities $T = 1$, $T = 2.5$ and $T = 5$.
- (right): $\gamma_h = 0.3$, $\gamma_\sigma = 0.3$ $\rho = -0.5$, $\theta = 0.3$, $\mu = \ln(0.2)$, $\sigma_0 = 0.2$ and $F_0 = 100$, and $h = h_2$ for maturities $T = 1$, $T = 2.5$ and $T = 5$.

Thank you for your attention!

A question on volatility process

How to model volatility proecess?

Work based on **SCJLJL**[H]

Situation of the problem

[H] **SCJLJL**.

Experiments with S&P 500 Data

First problem: volatility is not directly observed on the market.
We have tested two classical strategies to obtain volatility signals.

- Historical volatility is estimated directly from high frequency records of prices.
- The second approach uses quoted vanilla option prices to obtain an estimation of the integrated local volatility.

Experiments with S&P 500 Data: A proposition of stochastic volatility model

In order to fit the non-stationary local regularity of volatility as measured on data, as well as of maintaining long-range dependence properties, we propose to model volatility process, *written in a risk-neutral* setting by:

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\ln(\sigma_t) = \theta\left(\mu - \ln(\sigma_t)\right) dt + \gamma_h d^{\diamond} B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \\ d\langle W, W^\sigma \rangle_t = \rho dt, \end{cases}$$

where B_t^h is an multifractional Brownian motion and W and W^σ are two Brownian motions.

In addition, as we will show from numerical experiments, this model yields shapes of the smile at maturity T.

Experiments with S&P 500 Data: First method

Historical volatility is estimated directly from high frequency records of prices. Our raw data are minute quotes of the S&P 500, recorded from 2/2/2012 to 7/23/2012 (47 748 samples). To estimate the historical volatility:

1. Compute the returns by taking logarithms of differences
2. process the data to remove the high frequency market microstructure noise (low-pass filter)
3. groupe the samples into blocks corresponding to a time period of four hours.
4. the volatility attached to a block is then estimated as the standard deviation of the filtered samples contained in this block

Experiments with S&P 500 Data

In both cases, the volatility appears to be highly irregular.

Figure: S&P 500 minute data (left) and estimated volatility for time periods of four hours (right).

Note that our estimated historical volatility bears some resemblance with the ones displayed *e.g.* in **ADBE**[I].

[I] **ADBE**.

Experiments with S&P 500 Data: Estimation of the local regularity of the volatility

- Using the increment ratio statistic of **BS10**[J],

we determine an estimation of the functional parameter h. This gives the results displayed on Figure 6 below.
As we shall use this regularity as an input for our model below, we need an analytical expression for it. We have thus regressed the raw regularity using a simple sine function, also shown on Figure 6 below.
We denote this regressed function $\widehat{h_{VolSP}}$ and it will be used in our numerical experiments.

[J] **BS10**.

Experiments with S&P 500 Data

Figure: Estimated regularity of the volatility of the S&P 500 minute data (blue) and its regression (green).

Conclusion

Local regularity estimated in this way on the volatility of the S&P 500 is clearly not constant in time. It seems to oscillate with a period of roughly six weeks, and ranges approximately between 0.2 and 0.8.

Recalls about the Wiener-Itô theorem

$(e_n)_{n\in\mathbb{N}}$ constitutes an orthonormal basis of $(L^2(\mathbb{R}),\lambda)$ and the eigen vector family of the operator A.
Let (L^2) be the space $L^2(\Omega,\mathcal{G},\mu)$ where $\mathcal{G}$ is the σ-field generated by $(<.,f>)_{f\in L^2(\mathbb{R})}$.

Theorem (Wiener-Itô)

For every r.v Φ of (L^2) there exists a unique sequence of functions f_n in $\widehat{L}^2(\mathbb{R}^n)^{\mathbb{N}}$ such that Φ can be decomposed as

$$\Phi=\sum_{n=0}^{+\infty} I_n(f_n),$$

where

- $\widehat{L}^2(\mathbb{R}^n)$ *denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$*
- $I_n(f)$ *denotes the n^{th} multiple Wiener-Itô integral of f defined by*

$$I_n(f):=\int_{\mathbb{R}^n} f(t)dB^n(t)=n!\int_{\mathbb{R}}\Big(\int_{-\infty}^{t_n}\cdots\Big(\int_{-\infty}^{t_2} f(t_1,\cdots,t_n)dB(t_1)\Big)dB(t_2)\cdots dB(t_n)\Big),$$

Moreover we have the isometry $E[\Phi^2]=\sum_{n=0}^{+\infty} n!\,||f_n||^2_{L^2(\mathbb{R}^n)}$.

Definition (second quantization operator)

Recall that $A := -\frac{d^2}{dx^2} + x^2 + 1$. *For any* $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ *satisfying the condition* $\sum_{n=0}^{+\infty} n!\ |A^{\otimes n} f_n|_0^2 < +\infty$, *define the element* $\Gamma(A)(\Phi)$ *of* (L^2) *by*

$$\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n),$$

where $A^{\otimes n}$ *denotes the* n^{th} *tensor power of the operator* A.

Remark

The operator $\Gamma(A)$ *is densely defined on* (L^2) *and shares a lot of properties with the operator* A.

Let us denote $||\varphi||_0^2 := ||\varphi||_{(L^2)}^2$ for any random variable φ in (L^2). The space of Hida distributions is defined in a way analogous to the one that allowed to define the space $\mathscr{S}'(\mathbb{R})$:

Space of Hida distributions

Define

- The family of norms $(|| \ ||_p)_{p\in\mathbb{Z}}$ by, $\forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\mathrm{om}(\Gamma(A)^p)$,

$$||\Phi||_p := ||\Gamma(A)^p \Phi||_0 = ||\Gamma(A)^p \Phi||_{(L^2)}.$$

- $\forall p$ in $\mathbb{N}$, $\quad (\mathcal{S}_p) := \{\Phi \in (L^2) : \ \Gamma(A)^p\Phi$ exists and belongs to $(L^2)\}$,
- $\forall p$ in $\mathbb{N}$, $\quad (\mathcal{S}_{-p}) := \overline{(L^2)}^{|| \ ||_{-p}}$.

Definition (Test functions and Hida distributions spaces)

We call space of stochastic test functions the space $(\mathcal{S})$ defined as being the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and space of Hida distributions the space $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$.

Remark

This means that we have the equalities

- $(\mathcal{S}) = \underset{p\in\mathbb{N}}{\cap}(\mathcal{S}_p)$ *and* $(\mathcal{S})^* = \underset{p\in\mathbb{N}}{\cup}(\mathcal{S}_{-p})$
- *that convergence in $(\mathcal{S})$ (resp. in $(\mathcal{S})^*$) means convergence in $(\mathcal{S}_p)$ for every p*

As previously one can show that, for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $\mathcal{S}_p$ is $(\mathcal{S}_{-p})$. Thus we will write $(\mathcal{S}_{-p})$, in the sequel, to denote the space $(\mathcal{S}_p)^*$.
We have

$$\boxed{\bigcap_{p\in\mathbb{N}} (\mathscr{S}_p) = (\mathscr{S}) \subset (L^2) \subset (\mathcal{S})^* = \bigcup_{p\in\mathbb{N}} (\mathcal{S}_{-p}) = \bigcup_{p\in\mathbb{N}} \overline{L^2(\mathbb{R})}^{\|\ \|_{-p}}}$$

Remark

- *$(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$ and we will note $<\,,\,>$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$.*
- *If Φ belongs to (L^2) then we have the equality*
  $<\!\!\Phi,\varphi\!\!> \;= <\Phi,\varphi>_{(L^2)} = \mathbb{E}[\Phi\,\varphi], \quad \forall\varphi\in(\mathscr{S}).$

1.5

Bibliogenerale

- First point...
- Second point which is mentioned in **NuTa06**