A truncated two-scales realized volatility estimator

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Conference on Ambit Fields and Related Topics, Aarhus

1. Introduction

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- A common practice in the analysis of high-frequency financial data is to estimate daily volatility using the classic realized volatility estimator assuming that asset prices follow a continuous stochastic model and are directly observed.
- Important in asset pricing, risk management and portfolio allocation.
- Log-price process: $(y_t, t \in [0,1])$ with dynamics

$$dy_t = a_t dt + \sigma_t dB_t, \qquad y_0 \in \mathbb{R},$$

  B is a standard Brownian motion, $[0,1]$ represents 1 day
- Observations: $y_i = y_{t_i}$, where $0 = t_0 < t_1 < \cdots < t_m = 1$,

$$h = t_i - t_{i-1} = \frac{1}{m}, \qquad m \text{ large}$$

- Goal: Estimate the integrated volatility: $\int_0^1 \sigma_t^2 dt$ from the observations

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$$\widehat{\sigma}^2_{\mathrm{RV}} = \sum_{i=1}^{m} (y_i - y_{i-1})^2$$

- Consistent estimator of the integrated volatility: as $m \to \infty$

$$\boxed{\widehat{\sigma}^2_{\mathrm{RV}} \xrightarrow{\mathrm{P}} \int_0^1 \sigma_t^2 dt.}$$

- Moreover, it satisfies the following CLT: as $m \to \infty$

$$\boxed{m^{1/2}\left(\widehat{\sigma}^2_{\mathrm{RV}} - \int_0^1 \sigma_t^2 dt\right) \xrightarrow{\mathcal{L}} N(0, c),}$$

  for some constant $c > 0$.

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- The presence of discontinuities has motivated modeling prices as a combination of a continuous and a jump process, but then it is more challenging to estimate the quadratic variation of the continuous part, which is typically the object of interest from an economic perspective.
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1. Power and multipower variation estimators: (Barndorff-Nielsen, Shephard:2004), (Corsi, Pirino, Reno:2010): finite activity jumps, (Barndorff-Nielsen, Shephard, Winkel:2006), (Woerner:2006), (Jacod:2008), (Jacod, Todorov:2014): infinite activity jumps.

2. Truncated realized volatility estimator (Mancini:2008, 2009): finite and infinite activity jumps.

$$\widehat{\sigma}^2_{\mathrm{TRV}} = \sum_{i=1}^m (y_i - y_{i-1})^2 \mathbf{1}_{\{(y_i - y_{i-1})^2 \leq r(h)\}},$$

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2. Theory

Truncated two-scales realized volatility estimator

- We introduce a novel realized volatility estimator that is consistent in the presence of both jumps and noise.
- In the spirit of Mancini we introduce a truncation technique based on a local average of intra-daily returns that allows to detect jumps when the price is contaminated by noise.
- We apply this technique to the two-scales realized volatility estimator to introduce the so called truncated two-scales realized volatility estimator.
- We establish consistency in the presence of finite or infinite activity jumps and noise. In case of finite activity jumps, we also establish its asymptotic distribution.
- A simulation study shows that it performs satisfactorily and out–performs the truncated realized volatility, the two-scales realized volatility, the bipower variation and the modulated bipower variation.

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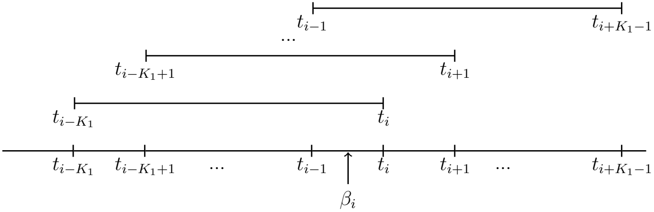
Jump signaling measure

- In order to detect a jump in a given interval $(t_{i-1}, t_i]$, we set

$$\beta_i = \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \left(x_j - x_{j-K_1} \right), \quad \text{for} \quad i = 1, \dots, m,$$

where $K_1 = K_1(m)$ satisfies $\lim_{m\to\infty} \frac{K_1}{m} = 0$ and $\lim_{m\to\infty} K_1 = \infty$.

- The β_i measure is a local average of overlapping returns:



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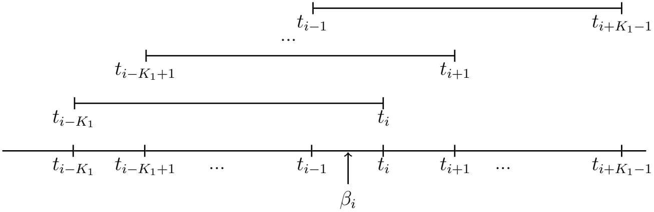
$$\beta_i = \frac{1}{K_1} \sum_{j=i}^{i+K_1-1} \left(x_j - x_{j-K_1}\right), \quad \text{for} \quad i = 1, \ldots, m,$$

where $K_1 = K_1(m)$ satisfies $\lim_{m\to\infty} \frac{K_1}{m} = 0$ and $\lim_{m\to\infty} K_1 = \infty$.

- The β_i measure is a local average of overlapping returns:



Jump signaling measure: finite activity jumps

Theorem 1

Suppose that

(1) σ and a are a.s. bounded on $[0, 1]$.

(2) $r(h)$ is a deterministic function such that

$$\lim_{h \to 0} r(h) = 0, \quad \lim_{h \to 0} \frac{\sqrt{\frac{\log \frac{1}{h}}{K_1}}}{r(h)} = 0, \text{ and } \lim_{h \to 0} \frac{\sqrt{K_1 h \log \frac{1}{h}}}{r(h)} = 0.$$

Then, for P-almost all ω, there exists $\overline{h}(\omega) > 0$ such that for all $h \leq \overline{h}(\omega)$ and $i = 1, \ldots, m$,

$$\mathbf{1}_{\{N_{i+K_1-1} - N_{i-K_1} = 0\}}(\omega) \leq \mathbf{1}_{\{|\beta_i| \leq r(h)\}}(\omega) \text{ and } \mathbf{1}_{\{|\beta_i| \leq r(h)\}}(\omega) \leq \mathbf{1}_{\{N_i - N_{i-1} = 0\}}(\omega).$$

Example: $K_1 = m^{\alpha_1}$, $r(h) = h^{\alpha_2}$, $0 < \alpha < 1$, $0 < \alpha_2 < \min\left(\frac{\alpha_1}{2}, \frac{1-\alpha_1}{2}\right)$.

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Modified TSRV estimator

- Recall the TSRV estimator (Zhang, Mykland, Aït-Sahalia:2005)

$$\widehat{\sigma}^2_{\text{TS}} = \frac{1}{K}\sum_{j=K}^{m}\left(x_j - x_{j-K}\right)^2 - \frac{m-K+1}{mK}\sum_{j=1}^{m}\left(x_j - x_{j-1}\right)^2,$$

where $K = cm^{2/3}$.

- We work with the following modified version of the TSRV

$$\widehat{\sigma}^2_{\text{MTS}} = \frac{1}{K}\sum_{j=K}^{m}(x_j - x_{j-K})^2 - \frac{1}{K}\sum_{j=K}^{m}(x_j - x_{j-1})^2.$$

- Both have the same asymptotic properties but $\widehat{\sigma}^2_{\text{MTS}}$ is easier to truncate.

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Truncated TSRV estimator

- The truncated TSRV realized volatility estimator (TTSRV)

$$\widehat{\sigma}^2_{\mathrm{TTS}} = \frac{1}{K}\sum_{j=K}^{m}(x_j - x_{j-K})^2 \mathbf{1}_{E_j} - \frac{1}{K}\sum_{j=K}^{m}(x_j - x_{j-1})^2 \mathbf{1}_{E_j},$$

where

$$E_j = \{|\beta_i| \leq r(h), i = j-K+1, \ldots, j\}.$$

- In the presence of finite or infinite activity jumps and noise, we show that this estimator estimates consistently the integrated volatility.
- When jumps are finite we show it has the same asymptotic distribution as the TSRV.
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Asymptotic properties: finite activity jumps

Consistency:

Theorem 2

Consider the assumptions of Theorem 1, a and σ a.s. continuous on $[0,1]$, and $\lim_{m\to\infty} \frac{K_1 \log m}{m^{1/3}} = 0$. Then as $m \to \infty$, $\widehat{\sigma}^2_{\text{TTS}} \xrightarrow{\text{P}} \int_0^1 \sigma_t^2 dt$.

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Asymptotic Normality:

Theorem 3

Consider the assumptions of Theorem 2, and that $\lim_{m\to\infty} \frac{K_1 \log m}{m^{1/6}} = 0$. Then as $m \to \infty$, $m^{1/6}\left(\widehat{\sigma}^2_{\text{TTS}} - \int_0^1 \sigma_t^2 dt\right) \xrightarrow{\text{stable}\mathcal{L}} \left(8c^{-2}\eta^4 + \frac{4}{3}c\int_0^1 \sigma_t^4 dt\right)^{1/2} N(0,1)$.

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3. Simulation study

Simulation study

- We assume a trading day is eight hours long and x_t is measured each second ($m = 28,800$)

$$x_t = \int_0^t \sigma_s dB_s + J_t + u_t,$$

and

$$d\sigma_s = \kappa(v - \sigma_s) + \tau\sqrt{\sigma_s} dW_s,$$

where $v = 9$, $\tau = 2.74$, $\kappa = 0.1$, and B and W are independent BM.
- Model 1 (finite activity jumps): J a compound Poisson process with $\lambda = 2$, and jump sizes iid $N(0, \xi^2)$.
- Model 2 (infinite activity jumps): J a variance gamma (VG) process, $J_s = d_1 G_s + d_2 \overline{W}_{G_s}$, $d_1 = -0.8$, $d_2 = 0.8$, G_s is a Gamma random variable with shape s/b and scale $b > 0$, $\overline{W}$ is a BM independent of B and W.
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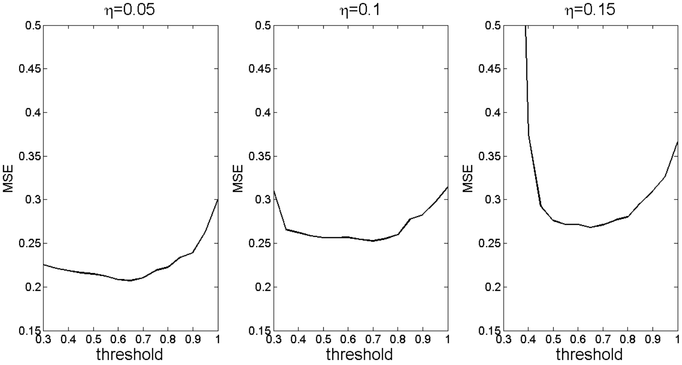
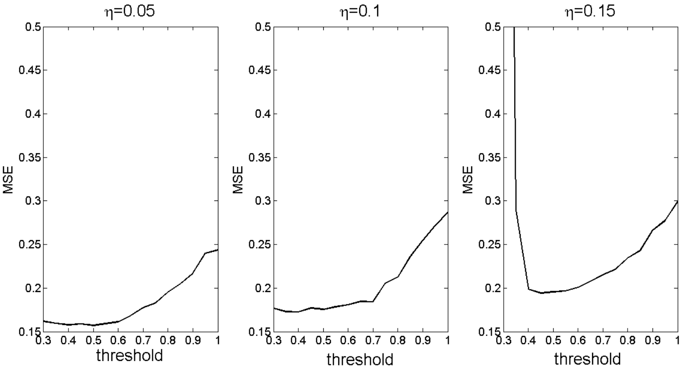
MSE of TTSRV: Model 1



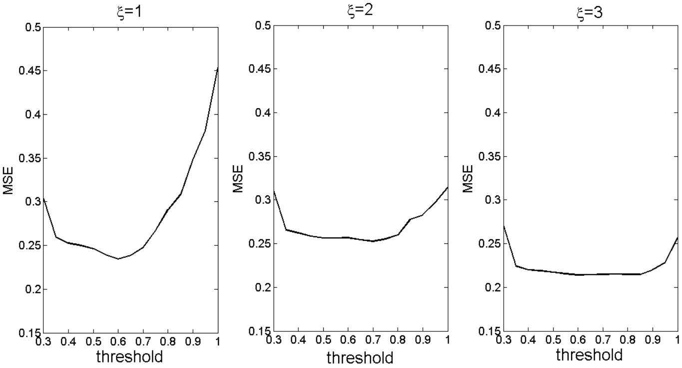
$\xi = 2$

MSE of TTSRV: Model 2



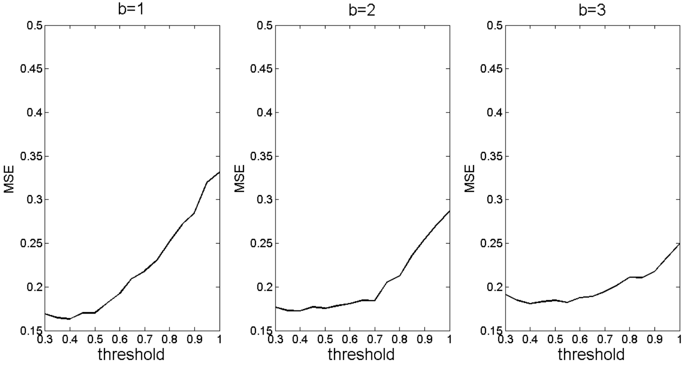
$b = 2$

MSE of TTSRV: Model 1



$\eta = 0.1$

MSE of TTSRV: Model 2



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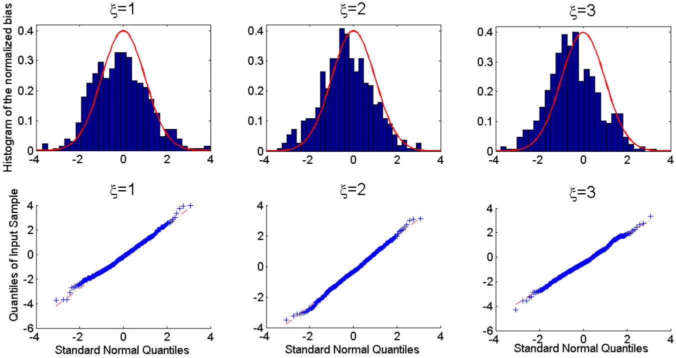
Histogram and normal qqplot: Model 1

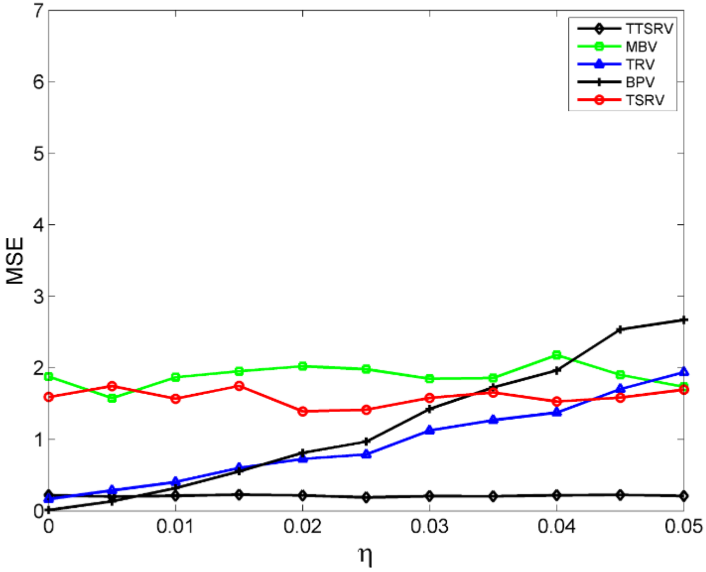


standardized estimation error:

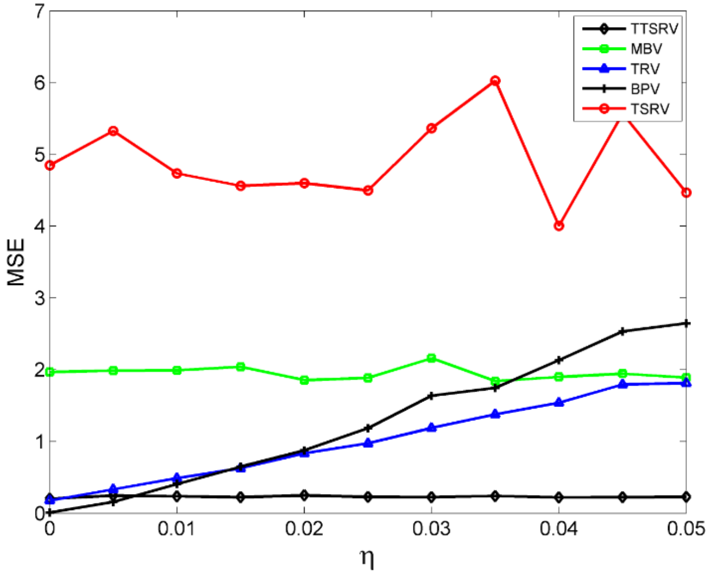
$$z = \frac{m^{1/6}\left(\widehat{\sigma}^2_{\mathrm{TTS}} - \int_0^1 \sigma_s^2 ds\right)}{\left(8c^{-2}\eta^4 + \frac{4}{3}c\int_0^1 \sigma_s^4 ds\right)^{1/2}}.$$

$r(h) = 0.75, \eta = 0.1$

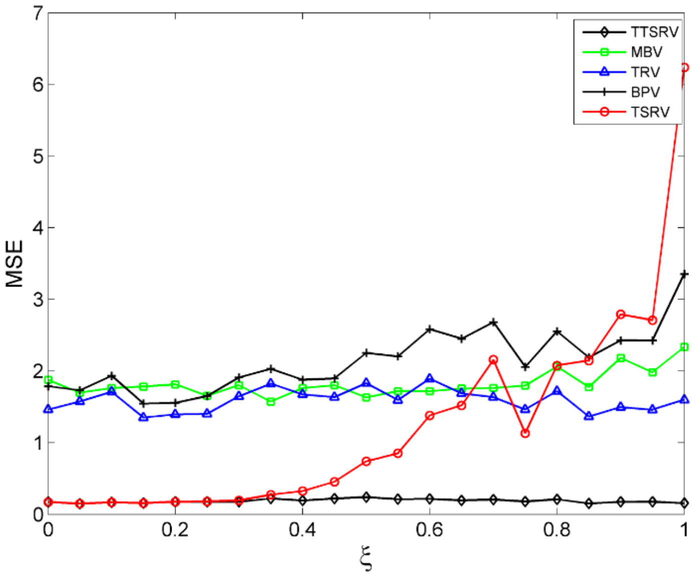
Comparison of MSE: Model 1



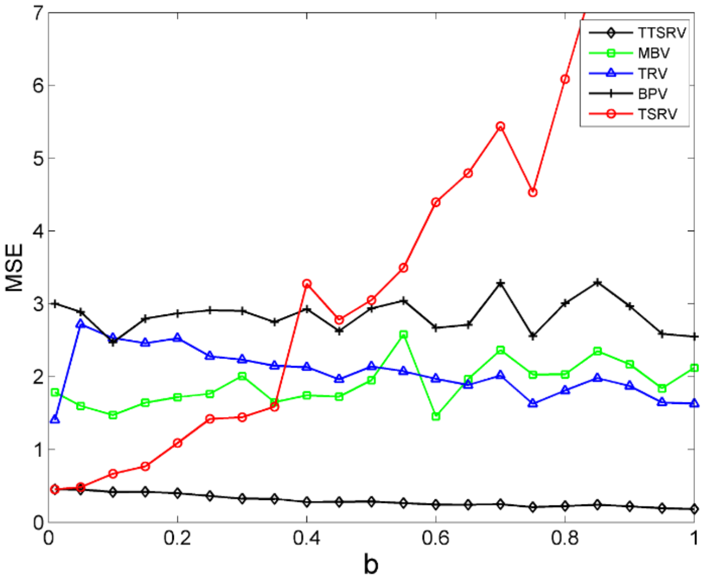
Comparison of MSE: Model 2



Comparison of MSE: Model 1



Comparison of MSE: Model 2



Thanks!