Representations and isomorphism identities for infinitely divisible processes

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Recall the Cameron-Martin Formula: Given a centered Gaussian process $G = (G_t)_{t\in T}$ over an arbitrary set T and a random variable ξ in L^2_G, the L^2-closure of the subspace spanned by G, we have for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\left[F\left((G_t + \phi(t))_{t\in T}\right)\right] = \mathbb{E}\left[F\left((G_t)_{t\in T}\right) e^{\xi - \frac{1}{2}\mathbb{E}\xi^2}\right] \qquad (1)$$

where $\phi(t) = \mathbb{E}(\xi G_t)$.

This formula has many applications, including SDEs and SPDEs driven by Gaussian random fields.

The C-M formula can also be viewed an isomorphism identity expressing a translated Gaussian process in terms of the untranslated process, but the latter is under the changed probability measure.

The set of all translation functions

$$\mathcal{H}_G = \{\phi : T \to \mathbb{R} : \phi(t) = \mathbb{E}(\xi G_t) \text{ for some } \xi \in L_G^2\}$$

forms a Hilbert space, called the Cameron-Martin space (or the reproducing kernel Hilbert space).

It is well-known that (1) does not extend to the Poissonian case.

Indeed, it is easy to see that if $Y = (Y_t)_{t \in [0,1]}$ is a Poisson process, then there is no function $\psi : [0,1] \to \mathbb{R}$, $\psi \not\equiv 0$ such that

$$\mathbb{E}\left[F\left((Y_t + \psi(t))_{t \in [0,1]}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t \in [0,1]}\right)\eta\right]$$

for all functionals F and some random variable $\eta \geq 0$ with $\mathbb{E}\eta = 1$.

We propose isomorphism identities based on random translations instead.

Namely, let $Y = (Y_t)_{t \in T}$ be a Poissonian infinitely divisible process over a general index set T. Let ν be the Lévy measure of Y on the path space $\mathbb{R}^T$ and assume that ν is σ-finite.

Let $Z = (Z_t)_{t \in T}$ be an arbitrary process, which is independent of the process Y, and whose distribution $\mathcal{L}(Z)$ on $\mathbb{R}^T$ is absolutely continuous with respect to ν, i.e., $\mathcal{L}(Z) \ll \nu$.

We will show that there exists a measurable functional $g : \mathbb{R}^T \mapsto \mathbb{R}_+$ with $\mathbb{E}g(Y) = 1$ such that for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$,

$$\mathbb{E}\left[F\left((Y_t + Z_t)_{t \in T}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t \in T}\right) g(Y)\right]. \tag{2}$$

What kind of functionals F can be of interest? A few examples:

- $F\left((Y_t)_{t\in T}\right) = f(Y_{t_1},\dots,Y_{t_n})$ cylindrical functional;
- $F\left((Y_t)_{t\in T}\right) = \sup_{t\in T} Y_t$ extremum;
- $F\left((Y_t)_{t\in T}\right) = \int_T |Y_t|^p\,\mu(dt)$ path integral;
- $F\left((Y_t)_{t\in[0,u]}\right) = \int_0^u \delta_y(Y_t)\,dt$ local time;
- $F\left((Y_t)_{t\in T}\right) = \int_0^\infty e^{-\eta_{t-}}\,d\xi_t$ exponential functional, where (η_t,ξ_t), $t\geq 0$ is a Lévy process, $T = R_1 \cup R_2$ the union of two disjoint copies of $\mathbb{R}_+$ and $Y_t = \eta_t$ if $t\in R_1$, $Y_t = \xi_t$ if $t \in R_2$.

Returning to our example of Poisson process $Y = (Y_t)_{t\in[0,1]}$, if we take

$$Z_t = \mathbf{1}_{[0,t]}(\eta)$$

where $\eta \in [0,1]$ is a random variable with absolutely continuous density f_η and independent of Y then (2) holds:

$$\mathbb{E}\left[F\left((Y_t + \mathbf{1}_{[0,t]}(\eta))_{t\in[0,1]}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t\in[0,1]}\right) g(Y)\right]$$

with $g(Y) = \lambda^{-1} \int_0^1 f_\eta(t)\, dY_t$ and λ being the rate of Y.

Any infinitely divisible process $X = (X_t)_{t \in T}$ can be written as

$$X \stackrel{d}{=} G + Y$$

where $G = (G_t)_{t \in T}$ and $Y = (Y_t)_{t \in T}$ are independent processes, G is centered Gaussian and Y is Poissonian infinitely divisible.

Combining isomorphis identities for Gaussian and Poissonian processes we get the identities for all infinitely divisible processes.

Isomorphism identities for Poissonian processes use Lévy measures.

What is the Lévy measure of a general infinitely divisible process?

Notation: Path space $\mathbb{R}^T = \{x : T \to \mathbb{R}\}$; $\mathcal{B}^T$ the cylindrical (product) σ-algebra of $\mathbb{R}^T$; 0_T the origin of $\mathbb{R}^T$.

Definition

A measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a Lévy measure if

(L1) for each $t \in T$

$$\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 \, \nu(dx) < \infty,$$

(L2) for every $A \in \mathcal{B}^T$

$$\nu(A) = \nu_*(A \setminus 0_T),$$

where ν_* denotes the inner measure.

Theorem (Lévy-Khintchine representation)

Let $X = (X_t)_{t \in T}$ be an infinitely divisible process. Then there exist a unique triplet (Σ, ν, b) consisting of a non-negative definite function Σ on $T \times T$, a Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and a function $b \in \mathbb{R}^T$ such that for every finite set $I \subset T$ and $a \in \mathbb{R}^I$

$$\mathbb{E} \exp i \sum_{t \in I} a_t X_t = \exp \Big\{ -\frac{1}{2} \langle a, \Sigma_I a \rangle + i \langle a, b_I \rangle + \int_{\mathbb{R}^T} (e^{\langle a, x_I \rangle} - 1 - i \langle a, [\![x_I]\!] \rangle)\, \nu(dx) \Big\} \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^I$ and $[\![\cdot]\!]$ denotes a truncation function.

Conversely, given Σ, a Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and $b \in \mathbb{R}^T$ there exists a unique in distribution infinitely divisible process $X = (X_t)_{t \in T}$ satisfying (3).

The following is possible but very rare ...

Theorem (Processes with not-σ-finite Lévy measures)

Let $Y = (Y_t)_{t \in T}$ be a Poissonian infinitely divisible process with Lévy measure ν. Then ν is not σ-finite if and only if T is uncountable and there is a version $\widetilde{Y} = (\widetilde{Y}_t)_{t \in T}$ of the process Y such that for every countable $T_0 \subset T$ there exist $t_1 \notin T_0$ and independent random variables ξ and η such that

(a) $\widetilde{Y}_{t_1} = \xi + \eta$;

(b) $(\widetilde{Y}_t, \xi, \eta : t \in T_0)$ *are jointly Poissonian infinitely divisible;*

(c) η *is non-degenerate and independent of* $(\widetilde{Y}_t, \xi : t \in T_0)$.

Remark

Intuitively, a Poissonian infinitely divisible process has a σ-finite Lévy measure if and only if outside some of countable $T_0 \subset T$ the process has no independent components to X_{T_0}.

Definition (separability in probability)

A stochastic process $Y = (Y_t)_{t \in T}$ is separable in probability if there exists a countable $T_0 \subset T$ such that for any $t \in T$ there is a sequence $\{s_n\} \subset T_0$ such that $Y_{s_n} \xrightarrow{P} Y_t$.

Corollary

A separable in probability Poissonian infinitely divisible process Y has a σ-finite Lévy measure.

Proof. Let T_0 be a separant for Y. Suppose to the contrary that ν is not σ-finite, so by above Theorem there is $t_1 \notin T_0$ such that (a)–(c) hold. By the separability, $\widetilde{Y}_{s_n} \xrightarrow{P} \widetilde{Y}_{t_1} = \xi + \eta$ for some $s_n \in T_0$. Therefore, $\xi + \eta$ is $\sigma(\widetilde{Y}_{T_0})$-measurable and still η is independent of $(\widetilde{Y}_t, \xi : t \in T_0)$. Hence, for any $u \in \mathbb{R}$

$$e^{iu(\xi+\eta)} = \mathbb{E}[e^{iu(\xi+\eta)} \mid \widetilde{Y}_{T_0}, \xi] = e^{iu\xi}\mathbb{E}[e^{iu\eta} \mid \widetilde{Y}_{T_0}, \xi] = e^{iu\xi}\mathbb{E}[e^{iu\eta}],$$

$|\mathbb{E}[e^{iu\eta}]| = 1$, so that η is deterministic. A contradiction. □

3. Representations and examples of Lévy measures of processes

A natural way to describe Lévy measures on path spaces is to view them as "laws of processes" defined on infinite measure spaces.

Definition

A collection of measurable functions $V = (V_t)_{t \in T}$ defined on a measure space $(S, \mathcal{S}, n)$ is said to be a representation of ν if all its finite dimensional "distributions" coincide with the corresponding projections of ν for all Borel sets that do not contain the origin.

A representation V is called exact if $n \circ V^{-1} = \nu$. Here V is viewed as a function from S into $\mathbb{R}^T$ given by $V(s)(\cdot) = V_{(\cdot)}(s)$.

Example (Lévy processes)

Let $Y = (Y_t)_{t\geq 0}$ be a Poissonian Lévy process determined by $\mathbb{E}e^{iuY_t} = e^{tK(u)}$, where K is the cumulant function given by

$$K(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iu[\![x]\!])\, \rho(dx) + iuc.$$

Then $V = (V_t)_{t\geq 0}$ defined on $(\mathbb{R}_+ \times \mathbb{R}, \lambda \otimes \rho)$ by

$$V_t(r, v) = \mathbf{1}_{\{t\geq r\}} v, \quad (r, v) \in \mathbb{R}_+ \times \mathbb{R}$$

is an exact representation of ν.

Example (Squared Bessel processes)

Let $Y = (Y_t)_{t\geq 0}$ be a squared Bessel process of dimension $\beta > 0$ starting from 0. If $\beta \in \mathbb{N}$, then $Y_t := \|B_t\|^2$, where B is a β-dimensional standard Brownian motion. In general, Y is defined as the unique solution of the stochastic differential equation

$$dY_t = 2\sqrt{Y_t}\, dW_t + \beta\, dt, \quad Y_0 = 0,$$

where W is a one dimensional standard Brownian motion. Shiga & Watanabe showed that squared Bessel processes are infinitely divisible and Pitman & Yor described the Lévy measures on $C(\mathbb{R}_+)$. We will adapt that characterization to our setting. Let $U_+ \subset C(\mathbb{R}_+)$ be defined by

$$U_+ := \{u : u(0) = 0,\ u_{|(0,t_0)} > 0,\ u_{|[t_0,\infty)} = 0 \text{ for some } t_0 > 0\}.$$

U_+ is a Borel subset of $C(\mathbb{R}_+)$, on which we consider the Itô measure n_+ of the Brownian positive excursions.

Example (Squared Bessel processes, cont.)

Let $L^a_\infty(u)$ denote the total accumulated local time of an excursion $u \in U_+$ at $a > 0$. Symbolically,

$$L^a_\infty(u) = \int_0^\infty \delta_a(u(t))\, dt.$$

Set $L^a_\infty(u) = 0$ when $a \leq 0$.

Then $V = (V_t)_{t \geq 0}$ defined on $(\mathbb{R}_+ \times U_+, \beta\lambda \otimes n_+)$ by

$$V_t(r, u) = L^{t-r}_\infty(u), \quad r \geq 0,\ u \in U_+.$$

is an exact representation of the Lévy measure ν of Y, the squared Bessel process of dimension β starting from zero.

Example (Feller diffusion)

We consider a Feller diffusion $Z = (Z_t)_{t \geq 0}$ without the drift term, which satisfies the stochastic differential equation

$$dZ_t = \sigma \sqrt{Z_t}\, dW_t, \quad Z_0 = a > 0,$$

where W is a one dimensional standard Brownian motion. Z is a time-scaled 0-dimensional squared Bessel process whose Lévy measure ν_0 was given by Pitman & Yor. We get that $V = (V_t)_{t \geq 0}$ defined on $(U_+, a\, n_+)$ by

$$V_t(u) = L_\infty^{4^{-1}\sigma^2 t}(u), \quad u \in U_+$$

is an exact representation of the Lévy measure of Z.

4. Lévy-Itô representations and transfer of regularity for Lévy measures

Theorem (Generalized Lévy-Itô representation)

Let $X = (X_t)_{t \in T}$ be a separable in probability infinitely divisible process with the generating triplet (Σ, ν, b). Assume that the probability space is rich enough to support independent of X standard uniform random variable. Then, given a representation $V = (V_t)_{t \in T}$ of ν defined on a σ-finite measure space $(S, \mathcal{S}, n)$, where $\mathcal{S}$ is (modulo n) countably generated, there exist a centered Gaussian process $G = (G_t)_{t \in T}$ with covariance Σ, an independent of G Poisson random measure N on $(S, \mathcal{S})$ with intensity n, such that for every $t \in T$

$$X_t = G_t + \int_S V_t(s) \left(N(ds) - \chi(V_t(s))\, n(ds)\right) + b(t) \quad a.s.$$

where χ is a fixed cut-off function.

Examples (integral representations)

(a) Lévy processes. Let $Y = (Y_t)_{t\geq 0}$ be a Poissonian Lévy process. Since $V_t(r,v) = \mathbf{1}_{\{t\geq r\}} v$, we get for every $t \geq 0$ a.s.

$$Y_t = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \mathbf{1}_{\{t\geq r\}} v \left(N(dr,dv) - \chi(\mathbf{1}_{\{t\geq r\}} v)\, dr \rho(dv) \right) + ct$$
$$= \int_0^t \int_{\mathbb{R}} v \left(N(dr,dv) - \chi(v)\, dr \rho(dv) \right) + ct,$$

where N is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with rate $\lambda \otimes \rho$.

(b) Squared Bessel processes. Let $Y = (Y_t)_{t\geq 0}$ be a squared Bessel process of dimension $\beta > 0$ starting from 0. Then

$$Y_t = \int_0^t \int_{U_+} L_\infty^{t-r}(u)\, N(dr,du)\,.$$

where N is s Poisson random measure on $\mathbb{R}_+ \times U_+$ with intensity $\beta\lambda \otimes n_+$. Therefore, a squared Bessel process Y is a mixed stochastic convolution.

Examples (integral representations, continue)

(c) Feller diffusion. Let $Z = (Z_t)_{t\geq 0}$ be a Feller diffusion starting from $a > 0$, as in a previous example. We have for every $t \geq 0$ a.s.

$$Z_t = \int_{U_+} L_\infty^{\kappa t}(u)\, N(du)\,.$$

where $\kappa = \sigma^2/4$ and N is s Poisson random measure on U_+ with intensity $a\, n_+$.

The transfer of regularity property for Lévy measures says that regularities of sample paths of Poissonian infinitely divisible processes are inherited by representations of their Lévy measures.

Theorem (Transfer of regularity)

Let $Y = (Y_t)_{t \in T}$ be a Poissonian infinitely divisible process with a σ-finite Lévy measure ν. Assume that paths of Y lie in a set U that is a subgroup of $\mathbb{R}^T$ under addition. Then ν has an exact representation with all paths in U. Therefore, both the distribution of Y and its Lévy measure are carried by the path space $(U, \mathcal{U})$, where $\mathcal{U} = \mathcal{B}^T \cap U$.

We continue isomorphic identities from the Introduction.

Theorem

Let $Y = (Y_t)_{t\in T}$ be a Poissonian infinitely divisible process with a σ-finite Lévy measure ν and given by its canonical spectral representation

$$Y_t = \int_{\mathbb{R}^T} x(t)[N(dx) - \chi(x(t))\nu(dx)] + b(t), \quad t \in T,$$

where N is a Poisson random measure with intensity ν. Let $Z = (Z_t)_{t\in T}$ be an arbitrary process independent of N such that $\mathcal{L}(Z) \ll \nu$. Put $q := \frac{d\mathcal{L}(Z)}{d\nu}$. Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\left((Y_t + Z_t)_{t\in T}\right) = \mathbb{E}\left[F\left((Y_t)_{t\in T}\right); N(q)\right] \tag{4}$$

Theorem (continue)

where

$$N(q) = \int_{\mathbb{R}^T} q(x)\, N(dx)\,.$$

Conversely, for any F as above,

$$\begin{aligned} &\mathbb{E}\left[F\left((Y_t)_{t\in T}\right);\, N(q) > 0\right] \\ &= \mathbb{E}\left[F\left((Y_t + Z_t)_{t\in T}\right)\left(N(q) + q(Z)\right)^{-1}\right] \end{aligned} \tag{5}$$

where $q(Z) = q\left((Z_t)_{t\in T}\right)$. Therefore, $\mathcal{L}(Y+Z)$ and $\mathcal{L}(Y)$ are equivalent provided $\nu\{x : q(x) > 0\} = \infty$.

The next identity is in terms of representations of Lévy measures.

Theorem

Let $Y = (Y_t)_{t\in T}$ be a Poissonian infinitely divisible process given by

$$Y_t = \int_S V_t(s)\Big[N(ds) - \chi(V_t(s))n(ds)\Big] + b(t)\,,$$

where $V = (V_t)_{t\in T}$ is a representation of the Lévy measure of Y defined on a σ-finite measure space $(S, \mathcal{S}, n)$, N is a Poisson random measure on $(S, \mathcal{S})$ with intensity n, and b is a shift function. Choose an arbitrary measurable function $q : S \mapsto \mathbb{R}_+$ such that $\int_S q(s)\, n(ds) = 1$. Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\int_S \mathbb{E}F\left((Y_t + V_t(s))_{t\in T}\right)\, q(s)\, n(ds) = \mathbb{E}[F\left((Y_t)_{t\in T}\right);\, N(q)]\,,$$

where

$$N(q) = \int_S q(s)\, N(ds)\,.$$

Theorem (Continue)

Conversely, for any F as above,

$$\mathbb{E}[F(\,(Y_t)_{t\in T}\,)\,;N(q)>0] \\ = \int_S \mathbb{E}\big[F\left(\left(Y_t + V_t(s)\right)_{t\in T}\right)\,;\,\left(N(q)+q(s)\right)^{-1}\big]\,q(s)\,n(ds)\,.$$

If $n\{s \in S : q(s) > 0\} = \infty$ *then*

$$\mathbb{E}[F\left((Y_t)_{t\in T}\right)] \\ = \int_S \mathbb{E}\Big[F\big((Y_t + V_t(s))\big)_{t\in T}\big);\,\left(N(q)+q(s)\right)^{-1}\Big]\,q(s)\,n(ds)\,.$$

Trying to understand the following isomorphism theorem inspired the present study:

Example (Dynkin isomorphism for permanental processes)

First we will recall some definition and facts about permanental processes that will be needed in the sequel. A positive real-valued stochastic process $Y = (Y_x)_{x \in E}$ over a set E is called a α-permanental process with kernel $(u(x,y) : x, y \in E)$ if for every $x_1, \dots, x_n \in E$ and $s_1, \dots, s_n \geq 0$

$$\mathbb{E} \exp \Big\{ -\sum_{j=1}^{n} s_j Y_{x_j} \Big\} = |I + US|^{-\alpha} \quad (6)$$

where U and S are $n \times n$-matrices, $U = (u(x_i, x_j) : 1 \leq i, j \leq n)$, $S = \mathrm{diag(s_1, \dots, s_n)}$, and $\alpha > 0$. The one dimensional marginal of Y_x is a gamma distribution with shape parameter α and mean $\alpha u(x,x)$ (in particular, it is exponential when $\alpha = 1$, or a χ^2-distribution when $\alpha \in \mathbb{N}/2$). Therefore, (6) can be viewed as a generalization of such distributions to the multivariate case.

Example (continue)

The importance of permanental processes comes from their connection to Markov processes. Eisenbaum & Kaspi proved that if $X = (X_t)_{t\geq 0}$ is a transient Markov process with a state space E and potential density $(u(x,y) : x, y \in E)$, then for every $\alpha > 0$ there exists a α-permanental process $Y^{(\alpha)} = (Y_x)_{x\in E}$ whose kernel is $(u(x,y) : x, y \in E)$ and $Y^{(\alpha)}$ is Poissonian infinitely divisible. Therefore

$$\mathbb{E}\exp\Big\{-\sum_{j=1}^{n} s_j Y_{x_j}\Big\} = \exp\left[\int_{\mathbb{R}_+^E}\left(e^{-\sum_{j=1}^{n} s_j\beta(x_j)} - 1\right)\alpha\nu(d\beta)\right],$$

where $x_1, \dots, x_n \in E$, $s_1, \dots, s_n \geq 0$, and $n \geq 1$. ν is the Lévy measure of the 1-permanental process. ν is σ-finite under a weak assumption that $Y^{(\alpha)}$ is separable in probability.

Example (continue)

The Dynkin Isomorphism Theorem says that for any measurable functional $F : \mathbb{R}^E \mapsto \mathbb{R}$

$$\mathbb{E}\tilde{\mathbb{E}}_a \left[F\left((Y_x + L_\infty^x)_{x \in E} \right) \right] = \alpha^{-1} u(a,a)^{-1} \, \mathbb{E}\left[F\left((Y_x)_{x \in E} \right); \; Y_a \right] , \tag{7}$$

where $(L_\infty^x)_{x \in E}$ is the process of the total accumulated local time at x of the associated Markov process X considered under probability measure $\tilde{\mathbb{P}}_a$. Under $\tilde{\mathbb{P}}_a$ the process X starts at a and is killed at its last visit to a.

One can show that $\mathcal{L}((L_\infty^x)_{x \in E}) \ll \nu$. Thus (7) is a special case of (4) above. Using (5) we also get

$$\mathbb{E}\left[F\left((Y_x)_{x \in E} \right) \right] = \alpha u(a,a) \, \tilde{\mathbb{E}}_a \mathbb{E}\left[F\left((Y_x + L_\infty^x)_{x \in E} \right) \left(Y_a + L_\infty^a \right)^{-1} \right] .$$

Example (Lévy processes)

Let $X = (X_t)_{t\geq 0}$ be a Lévy process such that $\mathbb{E}e^{iuX_t} = e^{tK(u)}$, where

$$K(u) = -\frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iu[\![x]\!])\,\rho(dx) + icu\,.$$

Let $q : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+$ be a measurable function such that $\int_{\mathbb{R}_+\times\mathbb{R}} q(r,v)\,dr\rho(dv) = 1$. Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\int_{\mathbb{R}_+\times\mathbb{R}} F\left(\left(X_t + \mathbf{1}_{\{r\leq t\}}v\right)_{t\geq 0}\right)\,q(r,v)\,dr\rho(dv)$$
$$= \mathbb{E}[F\left((X_t)_{t\geq 0}\right); g(X)]\,,$$

where $g(X) = \sum_{\{r>0:\,\Delta X_r\neq 0\}} q(r,\Delta X_r)$ and $\Delta X_r = X_r - X_{r-}$.

Example (Lévy processes, continue)

Conversely,

$$\mathbb{E}[F((X_t)_{t\geq 0}) \,; g(X) > 0]$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{E}\left[F\left(\left(X_t + \mathbf{1}_{\{r\leq t\}} v\right)_{t\geq 0}\right) ; \,(g(X) + q(r,v))^{-1}\right] q(r,v)$$
$$dr\rho(dv)\,.$$

Moreover, $g(X) > 0$ a.s. if $\int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{1}\{q(r,v) > 0\}\, dr\rho(dv) = \infty$.

We will only show an example how a representation of Lévy measure leads to series representation of the processes.

Example (Feller diffusions)

Let $Z = (Z_t)_{t \in T}$ be a Feller diffusion starting from $a > 0$ and without a drift term. Recall that $V_t = L_\infty^{\kappa t}$, $t \geq 0$ is a representation of the Lévy measure of Z on $(S, n) = (U_+, a\, n_+)$. Let $R(u)$ denote the length of an excursion $u \in U_+$. It is well-known that

$$n_+\{u : R(u) > x\} = \frac{1}{\sqrt{2\pi}} x^{-1/2}.$$

Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be such that $f(x) = 0$ only for $x = 0$ and

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty f'(x) x^{-1/2}\, dx = 1.$$

Example (continue)

Since

$$\int_{U_+} f(R(u))\, n_+(du) = \int_0^\infty f'(x)\, n_+\{u : R(u) > x\}\, dx = 1,$$

$n^{(1)}(du) := f(R(u))\, n_+(du)$ is a probability measure on U_+. Let $(\xi_j)_{j\in\mathbb{N}}$ be an i.i.d. sequence of random elements in U_+ with the common distribution $n^{(1)}$ and let $(\Gamma_j)_{j\in\mathbb{N}}$ be a sequence of partial sums of i.i.d. mean-one exponential random variables independent of the sequence $(\xi_j)_{j\in\mathbb{N}}$. Then

$$Z_t \overset{d}{=} \sum_{j=1}^\infty L_\infty^{\kappa t}(\xi_j)\mathbf{1}\{f(R(\xi_j)) \le a\Gamma_j^{-1}\}, \quad t \ge 0$$

and the convergence holds also a.s. uniformly in t on finite intervals.

Example (continue)

Let us take $f(x) = \sqrt{\frac{\pi}{2}}(x \wedge 1)$ for concreteness. Then the above formula becomes

$$Z_t \overset{d}{=} \sum_{j=1}^{\infty} L_{\infty}^{\kappa t}(\xi_j) \mathbf{1}\{R(\xi_j) \wedge 1 \leq (2/\pi)^{1/2} a \Gamma_j^{-1}\}, \quad t \geq 0.$$

This formula says that a Feller diffusion is the series of randomly trimmed total accumulated local times taken at the level κt, $t \geq 0$ from an infinite sample of Brownian excursions. This sample is taken according to the density $(\pi/2)^{1/2}(R \wedge 1)$ with respect to n_+.

Thank you!