Contact distribution in a thinned Boolean model with power law radii

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A germ-grain model

A germ-grain model is a model of random compact sets in $\mathbb{R}^d$ whose union is a random closed set.

Each random compact set is a grain.

The union of the grains is a grain cover.

A germ-grain model is hard-core if, with probability 1, the grains are disjoint.

A spherical Boolean model

- N a homogeneous Poisson random measure on $\mathbb{R}^d$ with intensity λ.
- Points $(\mathbf{X}_i)$ of N are marked by i.i.d. positive random vectors (R_i, W_i) independent of N.
- $\mathbf{X}_i$ is the center of the ith ball;
- R_i is the radius of the ith ball;
- W_i is the weight of the ith ball.

The joint law of (R, W) on $(0, \infty) \times (0, \infty)$ is G.

The radius and the wieght may or may not be independent.

Each Poisson ball is a grain. Denote

$$F(\cdot) = G\big(\cdot \times (0, \infty)\big) ,$$

the marginal distribution of the radius of a Poisson grain.

A basic assumption:

$$\int_0^\infty r^d \, F(dr) < \infty \, .$$

Then: a.s., only finitely many Poisson balls intersect any compact set in $\mathbb{R}^d$.

Hence: the union of all Poisson balls is a closed set.

The spherical Boolean model is a germ-grain model.

- We consider the spherical Boolean model is a germ-grain model with regularly varying tails.

$$\bar{F}(r) = P(R > r) = r^{-\alpha} L(r)\,.$$

- $\alpha > d$, L slowly varying.
- This model was suggested by Kuronen and Leskela (2013) to create long-range dependent germ-grain models.

- The spherical Boolean model is a simple model.
- It is NOT a hard-core model.
- The hard-core property is needed in many applications.
- How can one modify the spherical Boolean model to guarantee the hard-core property?
- A common approach is to use thinning.

Thinning of germ-grain model means removing some of the grains.

The remaining grains must be disjoint.

Thinning of the spherical Boolean model is done according to the weights of the Poisson balls.

The balls with small weights get “killed”.

Rules of thinning

- The ith Poisson ball with center at $\mathbf{X}_i$ has weight W_i.
- The ith ball competes with the jth ball if they intersect.
- The ith ball wins competition with the jth ball if $W_i > W_j$.
- If $W_i = W_j$, both balls lose the competition.
- A Poisson ball is not removed only if it wins all of its competitions.

We will consider 4 different types of the joint distribution of the radius of a ball and its weight.

Type 1 $W = R$ a.s. The ball with the larger radius has a larger weight.

In this type of thinning "large balls are retained".

Type 2 $W = 1/R$ a.s. The ball with the smaller radius has a larger weight.

In this type of thinning "small balls are retained".

Type 3 W is independent of R, $W \sim U(0, 1)$.

In this type of thinning "random balls are retained".

Type 4 $W = 1$ a.s. All the weights are the same.

In this type of thinning "isolated balls are retained".

How does the type of thinning affect the resulting hard-core germ-grain model?

Kuronen and Leskela (2013) studied the asymptotic behaviour of the covariance function of the thinned grain cover.

- The original spherical Boolean model:

$$\Phi = \left\{ (\mathbf{X}_i, R_i, W_i),\ i = 1, 2, \ldots \right\}.$$

- The thinned germ-grain model:

$$\Phi_{\mathrm{th}} = \left\{ (\mathbf{X}_i, R_i, W_i) \in \Phi :\ W_i > W_j \right.$$

$$\left. \text{for every } \ (\mathbf{X}_j, R_j, W_j) \in \Phi,\ j \neq i,\ B_{\mathbf{X}_i}(R_i) \cap B_{\mathbf{X}_j}(R_j) \neq \emptyset \right\}.$$

The thinned grain cover:

$$G_{\mathrm{th}} = \bigcup_{(\mathbf{X}_i, R_i, W_i) \in \Phi_{\mathrm{th}}} B_{\mathbf{X}_i}(R_i)\,.$$

The covariance function:

$$\gamma(\mathbf{x}, \mathbf{y}) = P\big(\mathbf{x} \in G_{\mathrm{th}},\, \mathbf{y} \in G_{\mathrm{th}}\big) - P\big(\mathbf{x} \in G_{\mathrm{th}}\big) P\big(\mathbf{y} \in G_{\mathrm{th}}\big)\,,$$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

By isotropy, $\gamma(\mathbf{x}, \mathbf{y}) = \gamma(\|\mathbf{x} - \mathbf{y}\|)$. Then

$$\gamma(r) \sim c\bar{F}(r)r^d, \; r \to \infty$$

for the “large balls retained”, “random balls retained” and “isolated balls retained” thinning.

$$\gamma(r) \leq c_1 e^{-c_2 r^d}, \; r > 0$$

for the “small balls retained” thinning.

No long range dependence when “small balls are retained”, and long range dependence in the other cases.

- Thinned spherical Boolean models are complicated.
- Until now, only the first and second order characteristics of the thinned models have been studied.
- More complicated characteristics are known only for the original spherical Boolean model.
- We study the contact distribution for the thinned model.

- The contact distribution of a germ-grain model is a measure of the amount of the empty space.
- It is a distribution function H on $(0, \infty)$ such that

$$\bar{H}(r) = P\big(B_{\mathbf{0}}(r) \cap G_{\mathrm{th}} = \emptyset \big| \mathbf{0} \notin G_{\mathrm{th}}\big) .$$

- A related notion: the empty space function.
- It is a distribution function H_e on $[0, \infty)$ such that

$$\bar{H}_e(r) = P\big(B_{\mathbf{0}}(r) \cap G_{\mathrm{th}} = \emptyset\big) .$$

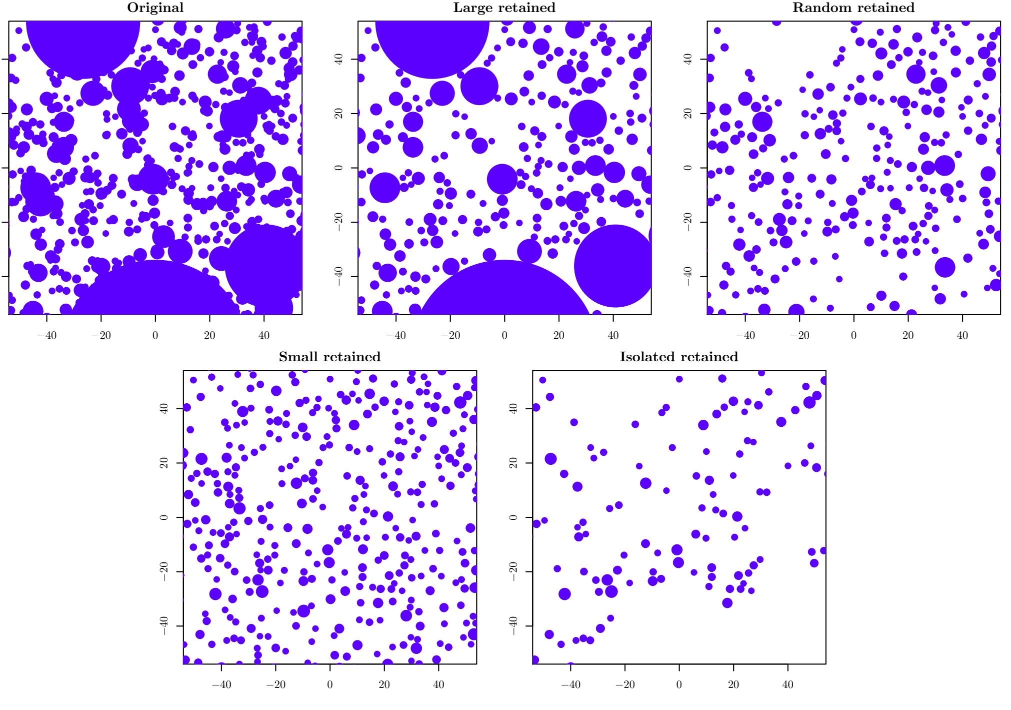
Question: What is the effect of thinning on the tail of the contact distribution?

For the original Boolean model,

$$\bar{H}_e(r) = P\Big(N\big(B_{\mathbf{0}}(r)\big) = 0\Big) = e^{-\lambda v_d r^d} \,,$$

v_d the volume of the unit ball in $\mathbb{R}^d$.

The tail of the contact distribution is decaying exponentially fast,



Original
Large retained
Random retained
Small retained
Isolated retained
−40
−20
0
20
40

- Thinned models have fewer balls.
- Does it mean that the have heavier tails of the contact distribution?
- It turns out that the contact distribution has power-like tails for the “large balls retained”, “random balls retained” and “isolated balls retained” thinning,
- The contact distribution has exponentiall tails for “small balls retained” thinning.

Theorem 1 For the "large balls retained" thinned spherical Boolean model,

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{\left(r^d \bar{F}(r)\right)^2} \leq \limsup_{r\to\infty} \frac{\bar{H}(r)}{\left(r^d \bar{F}(r)\right)^2} < \infty\,.$$

That is,

$$\bar{H}(r) \approx r^{-2(\alpha-d)}, \quad r \text{ large.}$$

Theorem 2 For the "isolated balls retained" thinned spherical Boolean model,

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{r^d \bar{F}(r)} \le \limsup_{r\to\infty} \frac{\bar{H}(r)}{r^d \bar{F}(r)} < \infty \,.$$

That is,

$$\bar{H}(r) \approx r^{-(\alpha-d)}, \quad r \text{ large.}$$

Theorem 3 For the "random balls retained" thinned spherical Boolean model,

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{\bar{F}(r)} \leq \limsup_{r\to\infty} \frac{\bar{H}(r)}{\bar{F}(r)} < \infty\,.$$

That is,

$$\bar{H}(r) \approx r^{-\alpha}, \quad r \text{ large.}$$

Theorem 4 For the “small balls retained” thinned spherical Boolean model,

$$\bar{H}(r) \leq e^{-cr^d}, \quad r \text{ large.}$$

In this model the tail of the contact distribution decays exponentially fast.

Intuition for the "large balls retained thinning":

For the event "no remaining balls intersect $B_{\mathbf{0}}(r)$" to occur two things are enough:

- There is one original Poisson ball big enough to cover the entire $B_{\mathbf{0}}(r)$.
- This ball "kills" all other Poisson balls intersecting $B_{\mathbf{0}}(r)$.
- There is another Poisson ball intersecting the first ball but not $B_{\mathbf{0}}(r)$, and larger than the first ball.

- The two events contribute about equally to the final probability.
- The Poisson process of balls assigns the mean measure of

$$cr^{-(\alpha-d)}, \quad r \;\text{ large}$$

  to the balls covering $B_{\mathbf{0}}(r)$.
- Hence the probability of the first event is about $cr^{-(\alpha-d)}$ for large r.