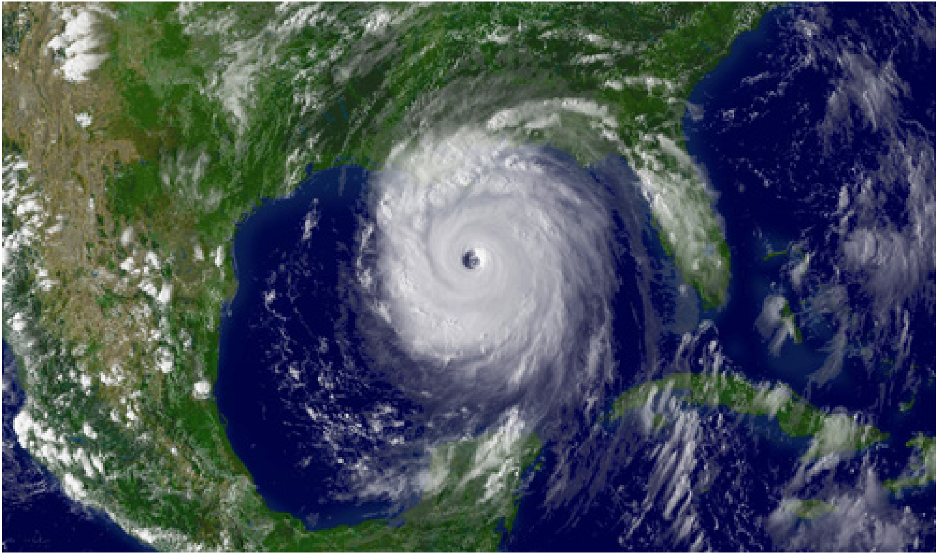
Stochastic Modelling of 2D Turbulence

O.E. Barndorff-Nielsen O. Sauri J. Schmiegel

Department of Mathematics
Aarhus University

Ambit Fields and Related Topics
Aarhus 2016



Outline

Outline

Ambit Stochastics

- General Ambit fields

$$Y_t(p) = \mu + \int_{A_t(p)} F(t,s,p,q)\sigma_s(q)L(\mathrm{d}s\mathrm{d}q) + \int_{B_t(p)} G(t,s,p,q)a_s(q)\mathrm{d}s\mathrm{d}q, \ (t,p) \in \mathbb{R} \times \mathbb{R}^d,$$

  - L a Lévy basis (infinitely divisible independently scattered random measure);
  - F, G deterministic functions;
  - σ, a random fields;
  - $A_t(p), B_t(p) \subseteq (-\infty, t] \times \mathbb{R}^d$.

- Stationary Ambit fields: Take (a, σ) stationary and
  - $F(t,s,p,q) = f(t-s, p-q)$ and $G(t,s,p,q) = g(t-s, p-q)$
  - $A_t(p) = A + (t,p)$ and $B_t(p) = B + (t,p)$, with $A, B \subseteq (-\infty, 0) \times \mathbb{R}^d$.

Ambit Stochastics

- Ambit processes: For a curve $\gamma(r) = (t_r, p_r) \in \mathbb{R} \times \mathbb{R}^d$

$$X_r = Y_{t_r}(p_r), \; r \in \mathbb{R}.$$

- Lévy semistationary

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s) a_s \mathrm{d}s + \int_{-\infty}^{t} f(t-s) \sigma_s \mathrm{d}L_s.$$

Ambit Stochastics

- Ambit processes: For a curve $\gamma(r) = (t_r, p_r) \in \mathbb{R} \times \mathbb{R}^d$

$$X_r = Y_{t_r}(p_r), \ r \in \mathbb{R}.$$

- Lévy semistationary

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s) a_s \mathrm{d}s + \int_{-\infty}^{t} f(t-s) \sigma_s \mathrm{d}L_s.$$

Ambit fields in 3D turbulence

- Key features in pure temporal turbulence:
  - Energy spectrum: $E(k) \propto k^{-5/3}$;
  - Intermittency on the velocity process;
  - Negative skewness;

Ambit fields in turbulence

- Non-parametric model: Ferrazzano and Küppelberg (2012), Brockwell et al. (2013)

$$Y_t = \mu + \int_{-\infty}^{t} f(t-s)\mathrm{d}L_s.$$

- Parametric model: Barndorff-Nielsen and Schmiegel (2008),Márquez and Schmiegel (2016)

$$Y_t = \mu + \beta \int_{-\infty}^{t} f(t-s)\sigma_s^2 \mathrm{d}s + \int_{-\infty}^{t} f(t-s)\sigma_s \mathrm{d}B_s.$$

Ambit fields in turbulence

- Non-parametric model: Ferrazzano and Küppelberg (2012), Brockwell et al. (2013)

$$Y_t = \mu + \int_{-\infty}^{t} f(t-s)\mathrm{d}L_s.$$

- Parametric model: Barndorff-Nielsen and Schmiegel (2008),Márquez and Schmiegel (2016)

$$Y_t = \mu + \beta \int_{-\infty}^{t} f(t-s)\sigma_s^2 \mathrm{d}s + \int_{-\infty}^{t} f(t-s)\sigma_s \mathrm{d}B_s.$$

Ambit fields in 3D turbulence

- Key features in spatio-temporal turbulence:
  - Energy spectrum: $E(k) \propto k^{-5/3}$;
  - Intermittency on the velocity process;
  - Negative skewness;
- Universality
  - Universal distribution (up to a time-change) on the velocity increments (Barndorff-Nielsen et al. (2004)).
  - The distribution of the logarithm of the energy dissipation does not depend on the Reynolds number (Hedevang and Schmiegel (2013)).

Ambit fields in 3D turbulence

- Hedevang and Schmiegel (2014) proposed a pure spatial model

$$Y(p) = \int_{\mathbb{R}^3} f(p-q)\sigma(q)L(\mathrm{d}q),$$

where $f \in \mathbb{M}_{3,d}(\mathbb{R})$ and L is $\mathbb{R}^d$-valued.

- Schmiegel (2005), Hedevang and Schmiegel (2013) introduced a model for the energy dissipation

$$\log \varepsilon(t,p) = \int_{A_t(p)} L(\mathrm{d}s\mathrm{d}q).$$

Stylized facts in 2D turbulence

- Double cascade for homogeneous and isotropic flows: $E(k) \propto k^{-5/3}$ and $E(k) \propto k^{-3}$;

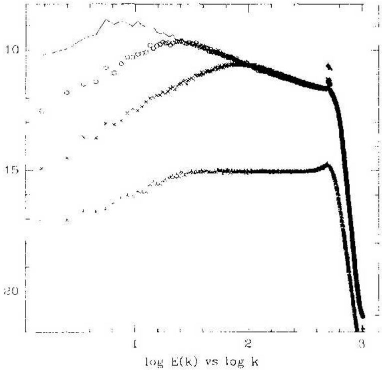


FIG. 1. Time evolution (increasing upward) of $E(k)$ for the 2048^2 run.

Figure: Typical behavior of the energy spectrum of 2D flows. Figure extracted from Smith and Yakhot (1993).

Stylized facts in 2D turbulence

- Vanishing divergence: If $Y_t(x,y) = \begin{bmatrix} Y_1(x,y) \\ Y_2(x,y) \end{bmatrix}$, then the *divergence* of Y is defined as
$$divY = \frac{\partial Y_1}{\partial x} + \frac{\partial Y_2}{\partial y}.$$
- Isotropic increments: A random field $(Y_t(p))_{t\in\mathbb{R},p\in\mathbb{R}^2}$ on $\mathbb{R}\times\mathbb{R}^2$ is said to have isotropic increments if for any $p\in\mathbb{R}^2$ it holds
$$Y_t(p) - Y_t(0) \overset{d}{=} R_\theta\left[Y_t(R_\theta^{-1}p) - Y_t(0)\right].$$
- Inttermitency in the vorticity but not in the increments.

Useful references

- Review of 2D turbulence:
  - Boffetta and Ecke (2012);
  - Tabeling (2002);
  - Rivera et al. (2001).

Outline

Stream functions

- A stream function $\psi_t(x,y)$ determines a velocity field $u_t(x,y)$:

$$u_t(x,y) = (-\partial_y \psi_t(x,y), \partial_x \psi_t(x,y)). \quad (1)$$

- There is no similar concept on 3D turbulence.
- If the velocity field $u_t(x,y)$ is obtained by a stream function, then it has null-divergence .
- The stream function can be found from vorticity using the following Poisson's equation

$$-\omega_t(x,y) = \partial_x^2 \psi_t(x,y) + \partial_y^2 \psi_t(x,y).$$

Intermittency in the vorticity can be reflected by the stream function.

Ambit-type stream functions

- To achieve null-divergence, isotropy and intermittency in the vorticity, we consider the class of processes

$$\psi_t(p) = \int_{H_t} F(t-s, \|p-q\|) V_s(q) L(\mathrm{d}s\mathrm{d}q),$$

where V is a random field and $H_t = (-\infty, t] \times \mathbb{R}^2$.

- ψ is a valid stream function only if it partially differentiable!

Ambit-type stream functions

- To achieve null-divergence, isotropy and intermittency in the vorticity, we consider the class of processes

$$\psi_t(p) = \int_{H_t} F(t-s, \|p-q\|) V_s(q) L(\mathrm{d}s\mathrm{d}q),$$

where V is a random field and $H_t = (-\infty, t] \times \mathbb{R}^2$.

- ψ is a valid stream function only if it partially differentiable!

Ambit-type stream functions

- If $V \equiv 1$, the partial differentiability can be obtained within the framework of Basse-O'Connor and Rosiński (2013).
- However, to get intermittency in the vorticity we require V to be stochastic.
- Fortunately V appears *linearly* on the stochastic integral.

Convergence of Ambit fields

- Let L be a *dispersive* Lévy basis with characteristic quadruplet $(\gamma_s(q), b_s(q), \nu_s(q;\mathrm{d}x), c(\mathrm{d}q\mathrm{d}s))_{s\in\mathbb{R},q\in\mathbb{R}^d}$. For $y, p \geq 0$, put

$$\Phi_L^p(y,s,q) := \sup_{|c|\leq 1} U(cy,s,q) + y^2 b_s^2(q) + \int_{\mathbb{R}} \left(|yx|^p \mathbf{1}_{\{|yx|>1\}} + |yx|^2 \mathbf{1}_{\{|yx|\leq 1\}}\right) \nu_s(q;\mathrm{d}x),$$

where

$$U(y,s,q) := \left| y\gamma_s(q) + \int_{\mathbb{R}} [\tau(yx) - y\tau(x)]\, \nu_s(q;\mathrm{d}x) \right|.$$

- From Basse-O'Connor et al. (2014), c.f. Rajput and Rosiński (1989) and Chong and Klüppelberg (2015),

$$\int_{\mathbb{R}\times\mathbb{R}^d} \varphi_n(s,q)\, L(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0,$$

if and only if

$$\int_{\mathbb{R}\times\mathbb{R}^d} \Phi_L^0(\varphi_n(s,q), s, q)\, c(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0.$$

Convergence of Ambit fields

- Let L be a *dispersive* Lévy basis with characteristic quadruplet $(\gamma_s(q), b_s(q), \nu_s(q;\mathrm{d}x), c(\mathrm{d}q\mathrm{d}s))_{s\in\mathbb{R},q\in\mathbb{R}^d}$. For $y,p\geq 0$, put

$$\Phi_L^p(y,s,q) := \sup_{|c|\leq 1} U(cy,s,q) + y^2 b_s^2(q) + \int_{\mathbb{R}} \left(|yx|^p \mathbf{1}_{\{|yx|>1\}} + |yx|^2 \mathbf{1}_{\{|yx|\leq 1\}} \right) \nu_s(q;\mathrm{d}x),$$

where

$$U(y,s,q) := \left| y\gamma_s(q) + \int_{\mathbb{R}} [\tau(yx) - y\tau(x)] \nu_s(q;\mathrm{d}x) \right|.$$

- From Basse-O'Connor et al. (2014), c.f. Rajput and Rosiński (1989) and Chong and Klüppelberg (2015),

$$\int_{\mathbb{R}\times\mathbb{R}^d} \varphi_n(s,q) L(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0,$$

if and only if

$$\int_{\mathbb{R}\times\mathbb{R}^d} \Phi_L^0(\varphi_n(s,q),s,q)\, c(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0.$$

Convergence of Ambit fields cont.

Proposition

Let

$$\varphi_n(s,q) := f_n(s,q)\,\sigma_s(q), \quad (s,q) \in \mathbb{R}\times\mathbb{R}^d,$$

where $(f_n)_{n\in\mathbb{N}}$ *is a sequence of deterministic functions, and* σ *a predictable random field which is bounded in* $\mathscr{L}^2(\Omega,\mathscr{F},\mathbb{P})$. *Then,* $\int_{\mathbb{R}}\int_{\mathbb{R}^d}\varphi_n(s,q)\,L(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0$ *if the following two conditions hold:*

1. $\int_{\mathbb{R}}\int_{\mathbb{R}^d} f_n(s,q)\,L(\mathrm{d}q\mathrm{d}s) \xrightarrow{\mathbb{P}} 0$;
2. $\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{E}\left[U(\varphi_n(s,q),s,q)\mathbf{1}_{\{|\sigma_s(q)|>1\}}\right]c(\mathrm{d}q\mathrm{d}s) \to 0.$

Differentiable Ambit fields

As a simple application of Basse-O'Connor and Rosiński (2013)

Proposition

Let L be a Lévy basis as above such that $\mathbb{E}[|L(A)|] < \infty$ for every $A \in \mathscr{B}_b(\mathbb{R} \times \mathbb{R}^d)$. Consider the Ambit field

$$Y_t(p) := \int_{H_t} F(t,p,s,q)\,\sigma_s(q)\,L(\mathrm{d}q\mathrm{d}s), \quad (t,p) \in \mathbb{R} \times \mathbb{R}^d,$$

with σ a predictable and $\mathscr{L}^2$-bounded. Suppose that the mapping $p_i \in \mathbb{R} \mapsto F(t,p_1,\ldots,p_i,\ldots,p_d,s,q)$ is absolutely continuous. If the mapping $p_i \mapsto \int_{H_t} \widetilde{\Phi}_L^1(\partial_{p_i} F(t,p,s,q),s,q)\,c(\mathrm{d}s\mathrm{d}q)$ is locally integrable, then the paths $p_i \mapsto X_t(p)$ are a.s. absolutely continuous with

$$\partial_{p_i} X_t(p) = \int_{H_t} \partial_{p_i} F(t,p,s,q)\,\sigma_s(q)\,L(\mathrm{d}q\mathrm{d}s). \tag{2}$$

When L is centered and square integrable, we only require the mapping $p_i \mapsto \int_{H_t} \partial_{p_i} F(t,p,s,q)^2\,c(\mathrm{d}s\mathrm{d}q)$ to be locally integrable.

A model for the velocity field

- For our model

$$\psi_t(p) = \int_{H_t} F(t-s, \|p-q\|) V_s(q) L(\mathrm{d}s\mathrm{d}q),$$

we get

$$u_t(x,y) = \int_{H_t} f(t-s, \|p-q\|) e(p-q) V_s(q) L(\mathrm{d}s\mathrm{d}q).$$

- $e(x,y) = (-y, x)$;
- $f(s,u) = u^{-1}\partial_u F(s,u)$.

A model for the velocity field

- The impact of e:

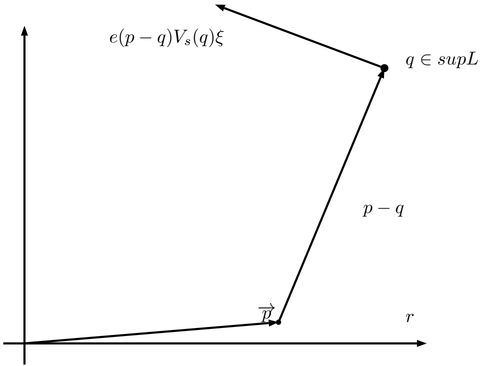


Figure: Rotation obtained by V and e.

Outline

Structure functions

- Given an isotropic velocity field, let

$$X_r := Y^1(r(1,0)), \; r \in \mathbb{R}. \tag{3}$$

- The structure function of X is defined by

$$S_n(r) := \mathbb{E}\left[(X_r - X_0)^n\right].$$

- Due to the double cascade, $S_2(r) \sim r^2$ near zero, and $S_2(r) \propto r^{2/3}$ outside of zero.
- Equivalently, the energy spectrum must satisfies that $E(k) \propto k^{-5/3}$ near zero, and $E(k) \propto k^{-3}$ outside of zero .

A model for the velocity field

- Inspired by Márquez and Schmiegel (2016), we consider

$$\psi(p) = \int_{\mathbb{R}^2} \phi_{\alpha,\beta,\lambda_1,\lambda_2}(\|p-q\|^2)L(\mathrm{d}q), \qquad p \in \mathbb{R}^2, \tag{4}$$

where $\phi_{\alpha,\beta,\lambda_1,\lambda_2} = \varphi_{\alpha,\lambda_1} * \varphi_{\beta,\lambda_2}$ with $\alpha, \beta > -1$, $\alpha+\beta > -3/2$, $\lambda_1 \vee \lambda_2 > 0$ and

$$\varphi_{\alpha,\lambda}(u) := u^{\alpha} e^{-\lambda u}, \; u \geq 0.$$

- We have that

$$S_2(r) \sim \begin{cases} c_{\alpha,\beta} r^{4(\alpha+\beta+1)} & \alpha+\beta \neq 3/4, -1 < \alpha+\beta < -1/2; \\ c_{\alpha,\beta} r^2 & -1/2 < \alpha+\beta. \end{cases}$$

A model for the velocity field

- Inspired by Márquez and Schmiegel (2016), we consider

$$\psi(p) = \int_{\mathbb{R}^2} \phi_{\alpha,\beta,\lambda_1,\lambda_2}(\|p-q\|^2)L(\mathrm{d}q), \qquad p \in \mathbb{R}^2, \tag{4}$$

  where $\phi_{\alpha,\beta,\lambda_1,\lambda_2} = \varphi_{\alpha,\lambda_1} * \varphi_{\beta,\lambda_2}$ with $\alpha, \beta > -1$, $\alpha + \beta > -3/2$, $\lambda_1 \vee \lambda_2 > 0$ and

$$\varphi_{\alpha,\lambda}(u) := u^{\alpha} e^{-\lambda u}, \ u \geq 0.$$

- We have that

$$S_2(r) \sim \begin{cases} c_{\alpha,\beta} r^{4(\alpha+\beta+1)} & \alpha+\beta \neq 3/4, -1 < \alpha+\beta < -1/2; \\ c_{\alpha,\beta} r^2 & -1/2 < \alpha+\beta. \end{cases}$$

Energy spectrum

- Recall that the energy spectrum behaves as

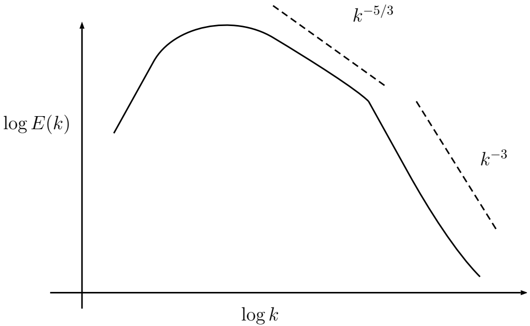


Figure: Typical behavior of the energy spectrum of 2D flows.

Energy spectrum

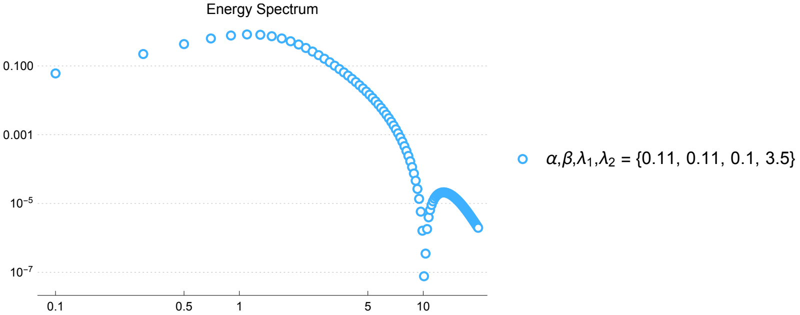


Figure: Energy Spectrum

Energy spectrum

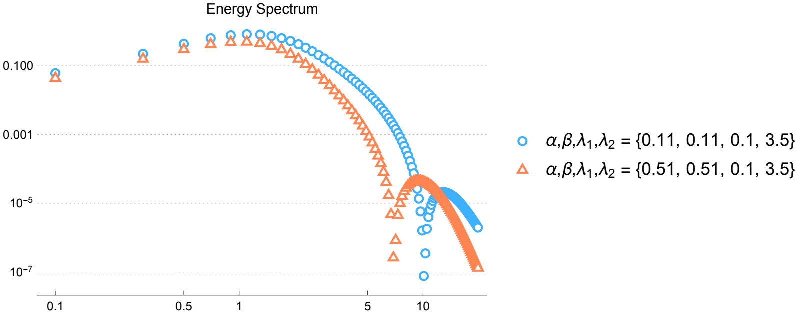


Figure: Energy Spectrum

Energy spectrum

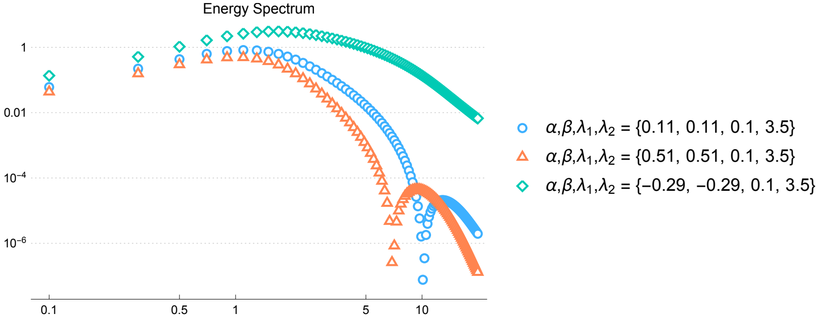


Figure: Energy Spectrum

Building the model from the Energy spectrum

- If we consider a general ambit-type stream function, i.e.

$$\psi(p) = \int_{\mathbb{R}^2} g(\|p-q\|^2)L(\mathrm{d}q), \qquad p \in \mathbb{R}^2.$$

its energy spectrum can be written as

$$E_Y(k) = \left[8\pi \int_0^\infty J_1(u\|k\|)g'(u^2)u^2\mathrm{d}u\right]^2, \; k \in \mathbb{R}^2.$$

- By the properties of the Hankel transform, we get that for $G(u) := g(u^2)$

$$G'(u) = \frac{1}{4\pi}\int_0^\infty J_1(ru)E_Y^{1/2}(r)r\mathrm{d}r.$$

Building the model from the Energy spectrum cont.

- The spectrum

$$E_{\mu,\nu,\lambda,l}(k) = (\lambda k)^{\mu}(\lambda\sqrt{k^2+l^2})^{\nu}K_{\nu}(\lambda\sqrt{k^2+l^2}),$$

according to (Hedevang and Schmiegel (2014)) behaves as

$$E_{\mu,\nu,\lambda,l}(k) \sim \begin{cases} (\lambda k)^{\mu} & k \ll l; \\ (\lambda k)^{\mu+2(\nu\wedge 0)} & l \ll k \ll \lambda^{-1} \\ (\lambda k)^{\mu+\nu-\frac{1}{2}}e^{-\lambda z}(1+\frac{c_{\nu}}{\lambda k}) & k \to \infty. \end{cases};$$

Building the model from the Energy spectrum cont.

- Inverting $E_{\mu,\nu,\lambda,l}(k)$, we get:

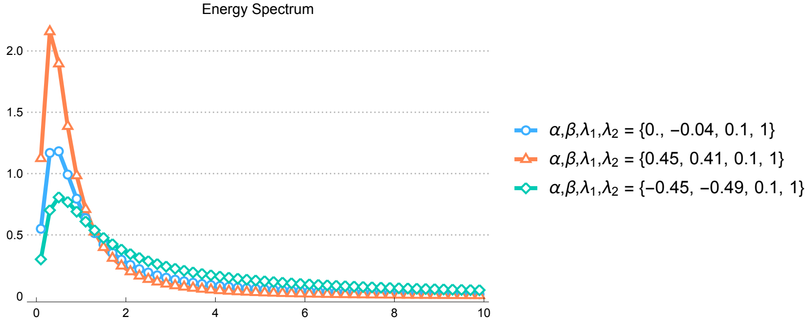


Figure: Kernel associated to $E_{\mu,\nu,\lambda,l}$.

Outline

Vorticity

- The concept of vorticity is rooted in that of circulation:

$$C_\rho\left(p;t\right) := \oint_{D(\rho,p)} u_t\left(\overrightarrow{q}\right) d\overrightarrow{q}.$$

- From this the *rotation* or *vorticity* of u is by definition determined from the circulation as

$$\omega_t\left(p\right) := \mathrm{rot} u_t\left(p\right) = \lim_{\rho \longrightarrow 0} \frac{1}{\pi\rho^2} C_\rho\left(p;t\right).$$

- In the smooth case:

$$\omega = \nabla \times u = \nabla^2 \psi.$$

Vorticity in Ambit Stochastics

- In our setting

$$C_{\rho}(p;t) = \int_{H_t} \varphi_{\rho}(t-s, p-q)\, \sigma_s(q)\, L(\mathrm{d}q\mathrm{d}s),$$

where

$$\varphi_{\rho}(t-s, p-q) := \rho \int_0^{2\pi} f(t-s, \|p-q+\rho\varepsilon(\theta)\|) \times \left\langle e(p-q+\rho\varepsilon(\theta)), \varepsilon^{\perp}(\theta) \right\rangle \mathrm{d}\theta,$$

with $\varepsilon(\theta) = (\cos(\theta), \sin(\theta))$ and $\varepsilon^{\perp} = (-\sin(\theta), \cos(\theta))$.

- Within our framework, the vorticity $\omega_t(p)$ is determined by the kernel F.

Stream function from vorticity

- In the smooth case, omitting the time-dependence

$$\frac{1}{4}\omega(p) = \int_{\mathbb{R}^2} h\left(\|p-q\|^2\right) V(q) L(\mathrm{d}q). \quad (5)$$

where

$$h(z) = F'(z) + zF''(z). \quad (6)$$

- If we start by assuming that the vorticity is given as in (5), we can obtain model for the vorticity F by solving (6):

$$g(z) = C_0 + [H(z) - h(z) + C] \log z,$$

where

$$H(z) = \int_0^z h(u) \mathrm{d}u.$$

Vistas

- Another important quantity in 2D turbulence is the determinant of the Hessian of the stream function:

$$\Lambda_t(x,y) = \partial_x^2 \psi_t(x,y) \partial_y^2 \psi_t(x,y) - [\partial_x \partial_y \psi_t(x,y)]^2 .$$

- Subject to incompressibility, this quantity uniquely determines the flow.
- Rivera et al. (2001) showed empirically that the distributions of Λ collapse after rescaling by the relative mean square of Λ.

Disitribution of Λ, the second chaos and infinite divisibility

- Within our framework, we get that in general

$$\Lambda_t(x,y) = \int_{H_t} f_1(t,s,p,q) V_s(q) L(\mathrm{d}s\mathrm{d}q) \int_{H_t} f_2(t,s,p,q) V_s(q) L(\mathrm{d}s\mathrm{d}q) - \left[\int_{H_t} f_3(t,s,p,q) V_s(q) L(\mathrm{d}s\mathrm{d}q)\right]^2.$$

- When $V \equiv 1$, the distribution of $\Lambda_t(x,y)$ belongs to the second Wiener chaos.
- Some characterizations for the infinite divisibility of $\Lambda_t(x,y)$ had been stablished in:
  - Griffiths (1970);
  - Bapat (1989);
  - Eisenbaum and Kaspi (2006).
- New characterizations by V. Rhode et al. in the poster session.
- Open problem: Characterization for the infinite divisibility of $\Lambda_t(x,y)$ in terms of f_i.

Disitribution of Λ, the second chaos and infinite divisibility

- Within our framework, we get that in general

$$\Lambda_t(x,y) = \int_{H_t} f_1(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\int_{H_t} f_2(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q) - \left[\int_{H_t} f_3(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\right]^2.$$

- When $V \equiv 1$, the distribution of $\Lambda_t(x,y)$ belongs to the second Wiener chaos.
- Some characterizations for the infinite divisibility of $\Lambda_t(x,y)$ had been stablished in:
  - Griffiths (1970);
  - Bapat (1989);
  - Eisenbaum and Kaspi (2006).
- New characterizations by V. Rhode et al. in the poster session.
- Open problem: Characterization for the infinite divisibility of $\Lambda_t(x,y)$ in terms of f_i.

Disitribution of Λ, the second chaos and infinite divisibility

- Within our framework, we get that in general

$$\Lambda_t(x,y) = \int_{H_t} f_1(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\int_{H_t} f_2(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q) - \left[\int_{H_t} f_3(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\right]^2.$$

- When $V \equiv 1$, the distribution of $\Lambda_t(x,y)$ belongs to the second Wiener chaos.
- Some characterizations for the infinite divisibility of $\Lambda_t(x,y)$ had been stablished in:
  - Griffiths (1970);
  - Bapat (1989);
  - Eisenbaum and Kaspi (2006).
- New characterizations by V. Rhode et al. in the poster session.
- Open problem: Characterization for the infinite divisibility of $\Lambda_t(x,y)$ in terms of f_i.

Disitribution of Λ, the second chaos and infinite divisibility

- Within our framework, we get that in general

$$\Lambda_t(x,y) = \int_{H_t} f_1(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\int_{H_t} f_2(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q) - \left[\int_{H_t} f_3(t,s,p,q)V_s(q)L(\mathrm{d}s\mathrm{d}q)\right]^2.$$

- When $V \equiv 1$, the distribution of $\Lambda_t(x,y)$ belongs to the second Wiener chaos.
- Some characterizations for the infinite divisibility of $\Lambda_t(x,y)$ had been stablished in:
  - Griffiths (1970);
  - Bapat (1989);
  - Eisenbaum and Kaspi (2006).
- New characterizations by V. Rhode et al. in the poster session.
- Open problem: Characterization for the infinite divisibility of $\Lambda_t(x,y)$ in terms of f_i.

Thank you!

References I

Bapat, R. B. (1989). Infinite divisibility of multivariate gamma distributions and m-matrices. *Sankhyā: The Indian Journal of Statistics, Series A (1961-2002) 51*(1), 73–78.

Barndorff-Nielsen, O. E., P. Blæsild, and J. Schmiegel (2004). A parsimonious and universal description of turbulent velocity increments. *The European Physical Journal B - Condensed Matter and Complex Systems 41*(3), 345–363.

Barndorff-Nielsen, O. E. and J. Schmiegel (2008). Time change, volatility, and turbulence. In *Mathematical control theory and finance*, pp. 29–53. Springer, Berlin.

Basse-O'Connor, A., S. Graversen, and J. Pedersen (2014). Stochastic integration on the real line. *Theory of Probability and Its Applications 58*, 355–380.

References II

Basse-O'Connor, A. and J. Rosiński (2013). Characterization of the finite variation property for a class of stationary increment infinitely divisible processes. *Stochastic Processes and their Applications 123*(6), 1871–1890.

Boffetta, G. and R. E. Ecke (2012). Two-dimensional turbulence. *Annual Review of Fluid Mechanics 44*(1), 427–451.

Brockwell, P. J., V. Ferrazzano, and C. Klüppelberg (2013). High-frequency sampling and kernel estimation for continuous-time moving average processes. *Journal of Time Series Analysis 34*(3), 385–404.

Chong, C. and C. Klüppelberg (2015). Integrability conditions for space-time stochastic integrals: Theory and applications. *Bernoulli 210*(4), 2190–2216.

Eisenbaum, N. and H. Kaspi (2006, 03). A characterization of the infinitely divisible squared gaussian processes. *Ann. Probab. 34*(2), 728–742.

References III

Ferrazzano, V. and C. Küppelberg (2012). Turbulence modeling by time-series methods.

Griffiths, R. C. (1970). Infinitely divisible multivariate gamma distributions. *Sankhyā: The Indian Journal of Statistics, Series A (1961-2002) 32*(4), 393–404.

Hedevang, E. and J. Schmiegel (2013). A causal continuous-time stochastic model for the turbulent energy cascade in a helium jet flow. *J. Turbul. 14*(11), 1–26.

Hedevang, E. and J. Schmiegel (2014). A Lévy based approach to random vector fields: With a view towards turbulence. *International Journal of Nonlinear Sciences and Numerical Simulation 15*(7 - 8), 411 – 435.

Márquez, J. U. and J. Schmiegel (2016). *Modelling Turbulent Time Series by BSS-Processes*, pp. 29–52. Cham: Springer International Publishing.

Rajput, B. S. and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields 82*(3), 451–487.

References IV

Rivera, M., X.-L. Wu, and C. Yeung (2001, Jul). Universal distribution of centers and saddles in two-dimensional turbulence. *Phys. Rev. Lett. 87*, 044501.

Schmiegel, J. (2005). Self-scaling of turbulent energy dissipation correlators. *Physics Letters A 337*(4–6), 342 – 353.

Smith, L. M. and V. Yakhot (1993, Jul). Bose condensation and small-scale structure generation in a random force driven 2d turbulence. *Phys. Rev. Lett. 71*, 352–355.

Tabeling, P. (2002). Two-dimensional turbulence: a physicist approach. *Phys. Rep* (362), 1–60.