Turbulence and Ambit Stochastics in Flatland

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Outline

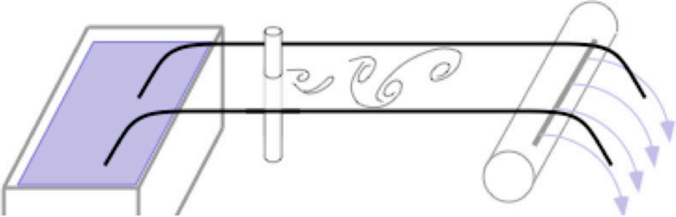
- 2-dimensional turbulence in nature and experiments
- Representations of 2-dimensional turbulence
- The double cascade scenario
- Ambit stochastics approach

3D: small scale structure from large scale motion

2D: large scale structure from small scale motion

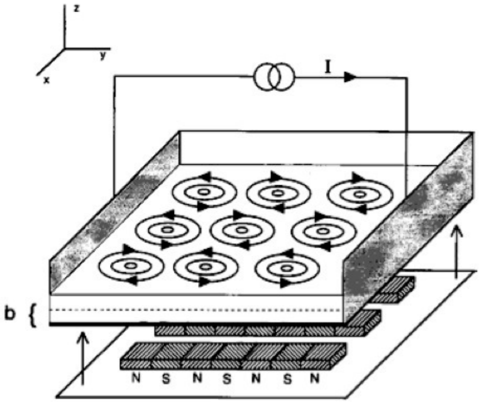
Soap film experiments

A. Lanotte, 2007

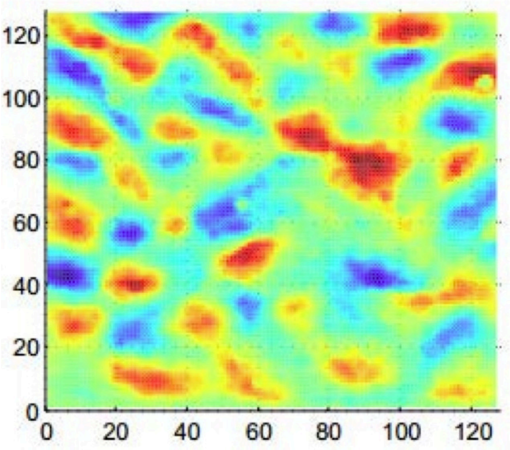


M. Gharib and P. Derango, Physica D37, 1989

Conducting flows



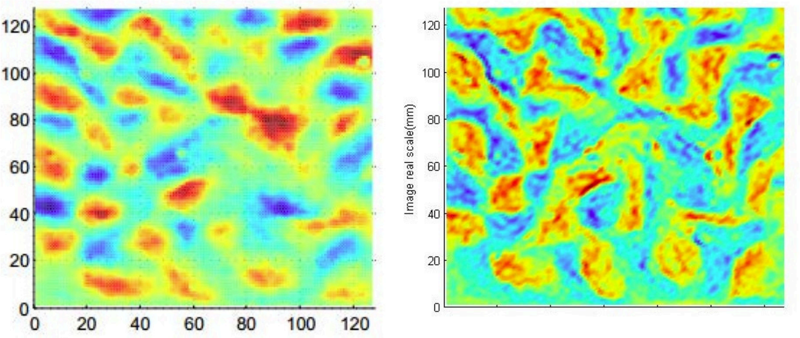
Velocity field: $\vec{v}(\vec{x}, t)$



D. Long 2015

Representations of 2-dimensional turbulence

Vorticity field: $\vec{\omega}(\vec{x},t) = \nabla \times \vec{v}(\vec{x},t) = \begin{pmatrix} 0 \\ 0 \\ \omega(\vec{x},t) \end{pmatrix}$



D. Long 2015

Stream function: $\Psi(\vec{x}, t)$

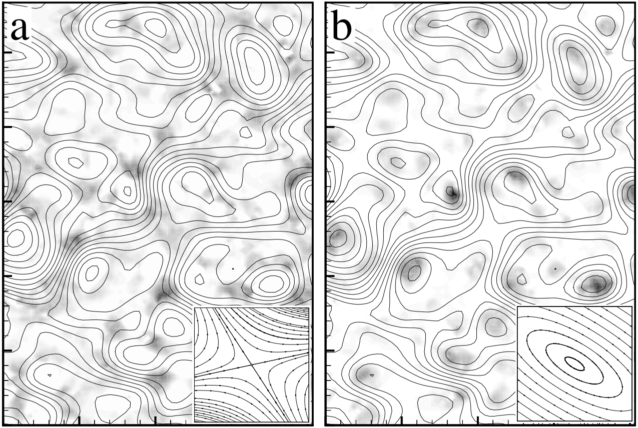
- velocity: $\vec{v}(\vec{x}, t) = \begin{pmatrix} \frac{\partial \Psi(x,y,t)}{\partial y} \\ -\frac{\partial \Psi(x,y,t)}{\partial x} \end{pmatrix}$
- path of a fluid element: streamlines $\Psi = const.$

Velocity gradient tensor: $A = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix}$

- determinant: $\Lambda = \det A = \frac{1}{4}\left(\omega^2 - \sigma^2\right)$
  enstrophy: ω^2
  strain rate: σ^2

- Weiss criterion:
  large positive $\Lambda \Rightarrow$ strong vorticity: center
  large negative $\Lambda \Rightarrow$ strong elongation: saddle point

Weiss criterion



M. Rivera and X. L. Wu 2000

The double cascade scenario

staionary 3D turbulence

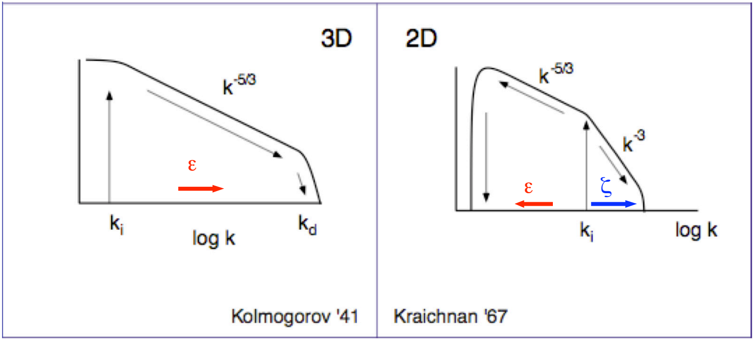
- no friction: energy is conserved
- small friction at boundaries (large scale)
- large internal friction (small scale)
- result: energy input cascades from large to small scales

stationary 2D turbulence

- no friction: energy and enstrophy is conserved
- large friction at boundaries (large scale)
- small internal friction (small scale)
- result: energy input cascades from small to large scales
- result: enstrophy cascades from large to small scales

Spectral representation

- Fourier transform $\vec{u}(\vec{k},t) = \mathbb{F}(\vec{v}(\vec{x},t))$
- Isotropic energy spectrum
  $E(k,t) = \frac{1}{2}\int_{|\vec{k}|=k} |\vec{u}(\vec{k},t)|^2 \mathrm{d}\Omega_k$



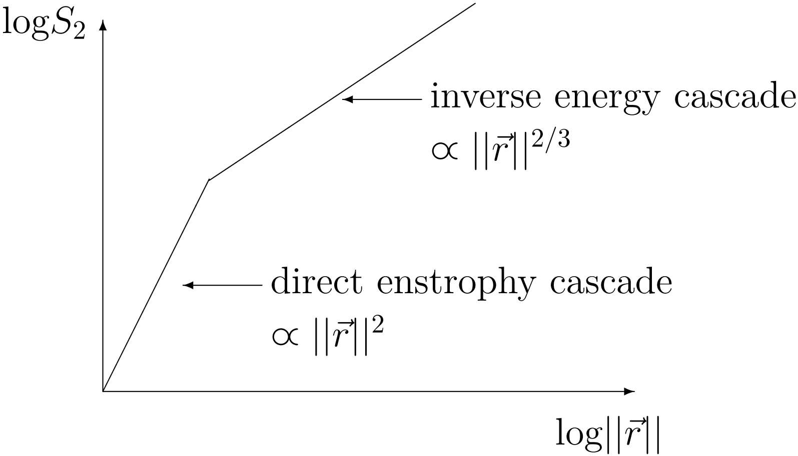
A. Lanotte 2007

Structure functions

$$S_n(\vec{r}) = \mathrm{E}\left\{ \left(\left(\vec{v}(\vec{r}) - \vec{v}(\vec{0}) \right) \cdot \frac{\vec{r}}{||\vec{r}||} \right)^n \right\}$$

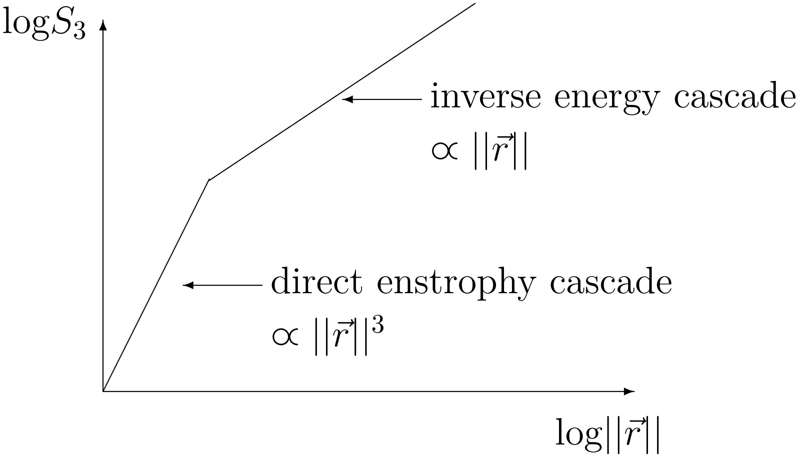
The double cascade scenario

second order structure function



The double cascade scenario

third order structure function



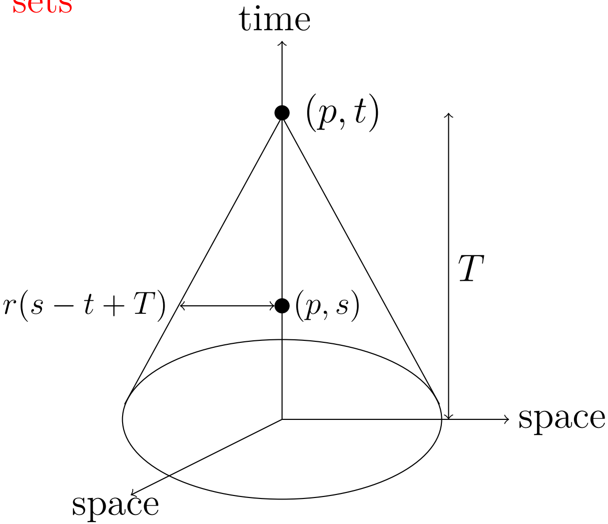
Ambit stochastics approach

homogeneous, isotropic, stationary model

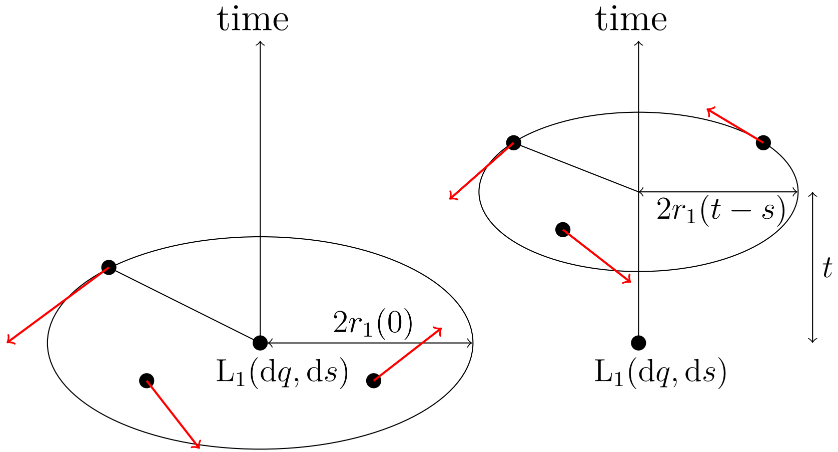
$$\vec{Y}(p,t) = \int_{A_1(t,p)} h_1(t-s, ||p-q||)\vec{e}_1(p,q)\mathrm{L}_1(\mathrm{d}q, \mathrm{d}s) + \int_{A_2(t,p)} h_2(t-s, ||p-q||)\vec{e}_2(p,q)\mathrm{L}_2(\mathrm{d}q, \mathrm{d}s)$$

- homogeneous and isotropic ambit sets A_1, A_2
- deterministic damping functions h_1, h_2
- independent homogeneous Lévy bases L_1, L_2
- $\vec{e}_1(p,q)$: attached to p, orthogonal to the vector connecting q with p and length depending on $||p-q||$
- $\vec{e}_2(p,q)$: attached to p, in direction of the vector connecting q with p and length depending on $||p-q||$

Ambit sets

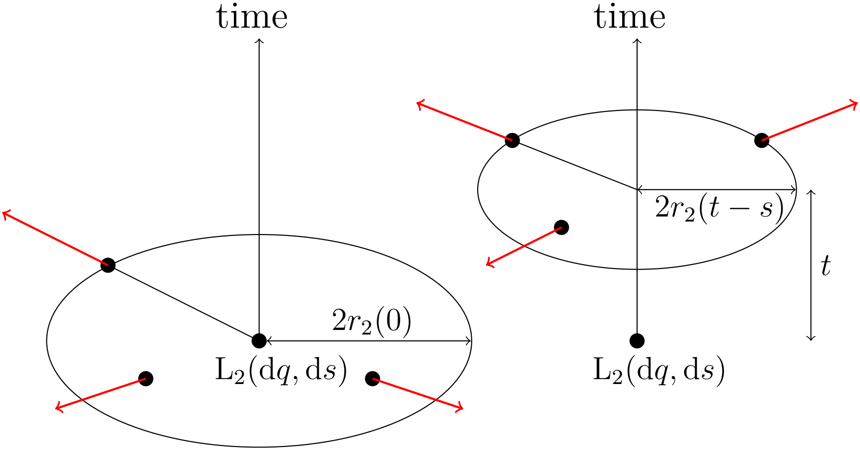


Interpretation: rotational part $\vec{e}_1$



Ambit stochastics approach

Interpretation: irrotational part $\vec{e}_2$

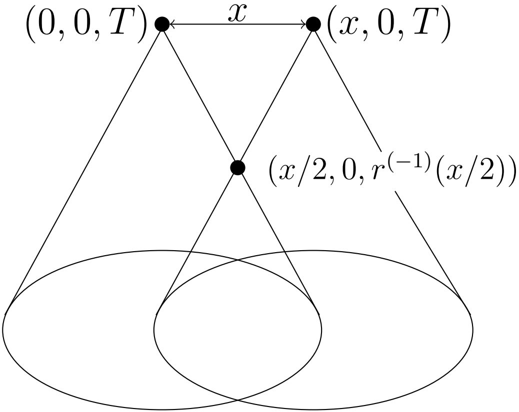


Simplification

$$\vec{Y}(p,t) = \int_{A_1(t,p)} h_1(t-s, ||p-q||)\vec{e}_1(p,q)(\mathrm{L}_1(\mathrm{d}q, \mathrm{d}s) + \int_{A_2(t,p)} h_2(t-s, ||p-q||)\vec{e}_2(p,q)(\mathrm{L}_2(\mathrm{d}q, \mathrm{d}s)$$

- i.i.d. Lévy bases L_1, L_2
- choose a cartesian (x, y) coordinate system
- rotational part $\vec{e}_1(p,q) = (q_2 - p_2, p_1 - q_1)$
- irrotational part $\vec{e}_2(p,q) = (p_1 - q_1, p_2 - q_2)$
- identical ambit sets $A_1(t,p) = A_2(t,p) = A(t,p)$, $r_1(s) = r_2(s) = r(s)$

Overlap of ambit sets



Second order structure function: $\mathrm{E}\{\mathrm{L}'\} = 0$

- assume differentiability of $S_2(x)$ at $x = 0$
- direct enstrophy cascade requires

$$S_2(x) \propto x^2 \ \text{ for } x \to 0$$

i.e. $S_2'(0) = 0$

- sufficient condition

$$h_1(s, r(T-s)) = h_2(s, r(T-s)) = 0$$

Ambit stochastics approach

Third order structure function: $\mathrm{E}\left\{\mathrm{L}'^{3}\right\} \neq 0$

- only the strain part contributes

Next talk by Orimar Sauri

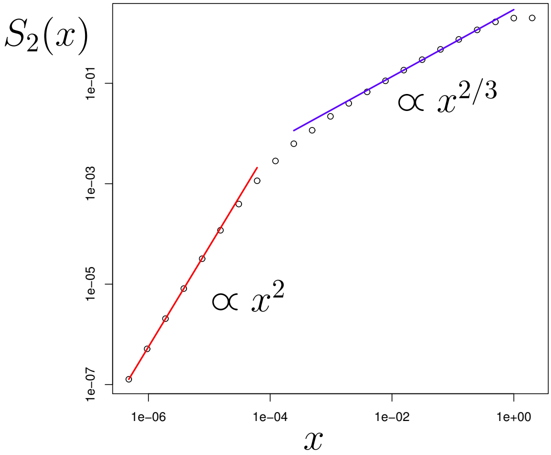
- purely spatial ambit set $A_1(p) = \mathbb{R}^2$
- only rotational part
- $h_1(x) = x^{2(a_1-1)} e^{-\lambda_1 x^2}(2\alpha_1 - \lambda_1 x^2)$
- $a_1 = 1/6$
- $S_2(x) \propto x^{2/3}$ for $x \to 0$
- modification:

$$h_1(s,x) = (x + x_0)^{2(a_1-1)} e^{-\lambda_1 (x+x_0)^2} (\lambda_1 (r(T-s)^2 - x^2))$$

Parameters

- $r(t) = T - t$, $T = 1$
- $h_1(s,x) = (x + x_0)^{2(a_1-1)} e^{-\lambda_1 (x+x_0)^2} (\lambda_1 (r(T-s)^2 - x^2)$
  $x_0 = 10^{-5}$, $\lambda_1 = 1$, $a_1 = 1/6$
- $h_2(s,x) = \beta (x + x_0)^{2(a_2-1)} e^{-\lambda_2 (x+x_0)^2} (\lambda_2 (r(T-s)^2 - x^2)$
  $\lambda_2 = 1$, $a_2 = 1/3$, $\beta = 0.1$

Second order structure function



Third order structure function