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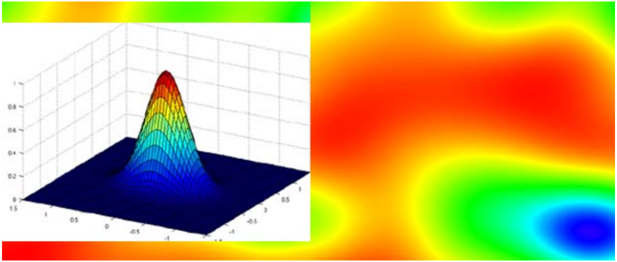

Limit theorems for functionals of stationary random fields

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Conference on *Ambit Fields and Related Topics*, Aarhus

Random field = Set of random variables indexed by $\mathbb{R}^d$

Aim:

Examine the asymptotic behavior of random variables

$$\int_{W_n} f(X(t))\, dt,$$

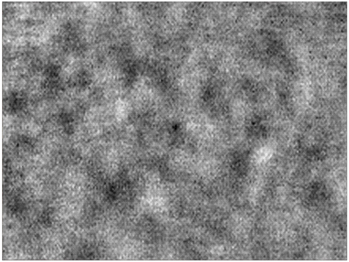
where $\{X(t)\}_{t\in\mathbb{R}^d}$ is a stationary random field and
$f : \mathbb{R} \to \mathbb{R}$ is a deterministic function

as W_n tends to the whole space $\mathbb{R}^d$.

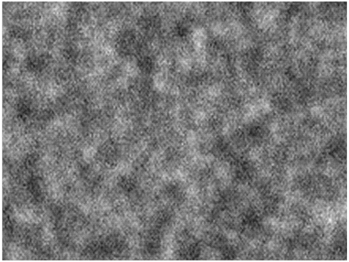
Overview

- Motivation
- Excursion sets of random fields and integral geometric functionals
- LTs: state of art
- CLT for the volume of excursion sets of stationary random fields
  - Second order quasi-associated fields
  - Examples: Shot noise, Gaussian case
  - **PA**- or **NA**-fields (possibly not second order!)
  - Examples: infinitely divisible, max- and α-stable fields
  - Multivariate CLT with a Gaussianity test
- FCLT
- Open problems

Motivation



Paper surface
(Voith Paper, Heidenheim)



Simulated Gaussian field
$\mathsf{E}X(t) = 126$
$r(t) = 491 \exp\left(-\frac{\|t\|_2}{56}\right)$

- Is the paper surface Gaussian?

Excursion sets

Let X be a measurable real-valued random field on $\mathbb{R}^d$, $d \geq 1$ and let $W \subset \mathbb{R}^d$ be a measurable subset. Then for $u \in \mathbb{R}$

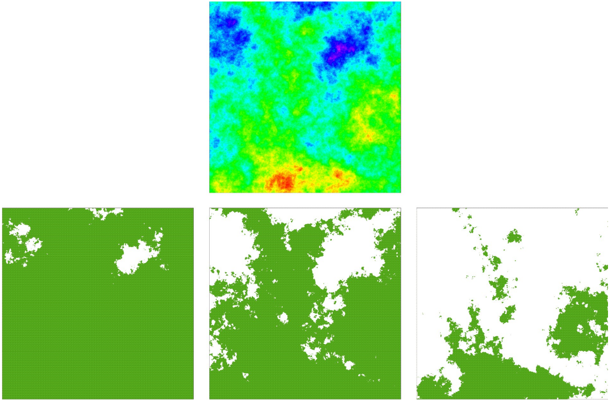
$$A_u(X, W) := \{t \in W : X(t) \geq u\}$$

is called the excursion set of X in W over the level u.

Volume of excursion set:

$$|A_u(X, W)| = \int_W f(X(t))\, dt, \qquad f(t) = \mathbb{1}\{t \geq u\},$$

here $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^d$ (cardinality of a set in $\mathbb{Z}^d$).



Centered Gaussian random field on $[0,1]^2$,
$r(t) = \exp(-\|t\|_2 / 0.3)$,
Levels: $u = -1.0, 0.0, 1.0$

LTs for integral geometric functionals of random fields

- Gaussian random fields
  - CLTs:
    - Stationary processes, $d = 1$: Belyaev & Nosko (1969); Cuzick (1976); Elizarov (1988); Kratz (2006)
    - Volume, $d \geq 2$: Ivanov & Leonenko (1989)
    - FCLT: Meschenmoser & Shashkin (2011)
- Non-Gaussian random fields
  - Integral functionals: Leonenko (1974), Bulinski & Zhurbenko (1976), Gorodetskii (1984) .
  - FCLT: Kampf & S. (2015)
  - Volume of excursion sets: Bulinski, S. & Timmermann (2012); Karcher (2012); Leonenko & Olenko (2014); Demichev (2013), Demichev & Olszewski (2015);

Growing sequence of observation windows

A sequence of compact Borel sets $(W_n)_{n\in\mathbb{N}}$ is called a Van Hove sequence (VH) if $W_n \uparrow \mathbb{R}^d$ with

$$\lim_{n\to\infty} |W_n| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(0)|}{|W_n|} = 0, \quad r > 0.$$

Theorem (Bulinskii & Zhurbenko, 1976)

Let $\{X(t)\}_{t\in\mathbb{R}^d}$ be a stationary, measurable random field fulfilling some α-mixing assumptions.
Let $f : \mathbb{R} \to \mathbb{R}$ be some measurable map such that $\{f(X(t))\}_{t\in\mathbb{R}^d}$ fulfills integrability assumptions.
Let $\{W_n\}_{n\in\mathbb{N}}$ be a VH-growing sequence of compact sets of $\mathbb{R}^d$.
Then

$$\frac{\int_{W_n} f(X(t))\,dt - |W_n|\cdot\mathbb{E}[f(X(0))]}{\sqrt{|W_n|}} \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

as $n\to\infty$, where

$$\sigma^2 = \int_{\mathbb{R}d} \operatorname{Cov}\left(f(X(0)), f(X(t))\right) dt.$$

Theorem (CLT for the volume of A_u at a fixed level $u \in \mathbb{R}$)

Let X be a strictly stationary random field satisfying some additional conditions and $u \in \mathbb{R}$ fixed. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$\frac{|A_u(X, W_n)| - \mathrm{P}(X(0) \geq u) \cdot |W_n|}{\sqrt{|W_n|}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \sigma^2(u)\right)$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \operatorname{cov}\left(\mathbb{1}\{X(0) \geq u\}, \mathbb{1}\{X(t) \geq u\}\right) dt.$$

Second order quasi-associated random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- square-integrable
- has a continuous covariance function $r(t) = \mathrm{Cov}(X(o), X(t))$, $t \in \mathbb{R}^d$
- $|r(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$
- $X(0)$ has a bounded density
- quasi-associated.

Then $\sigma^2(u) \in (0, \infty)$ (Bulinski, S., Timmermann (2012)).

Quasi-association

A random field $X = \{X(t), t \in \mathbb{R}^d\}$ with finite second moments is called quasi-associated if

$$|\mathrm{cov}\,(f\,(X_I)\,, g\,(X_J))| \leq \mathrm{Lip}\,(f)\,\mathrm{Lip}\,(g) \sum_{i \in I} \sum_{j \in J} |\mathrm{cov}\,(X\,(i)\,, X\,(j))|$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any Lipschitz functions $f : \mathbb{R}^{\mathrm{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\mathrm{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$.

Idea of the proof of the Theorem: apply a CLT for (BL, θ)-dependent stationary centered square-integrable random fields on $\mathbb{Z}^d$ (Bulinski & Shashkin, 2007).

(BL, θ)-dependence

A real-valued random field $X = \{X(t), t \in \mathbb{Z}^d\}$ is called (BL, θ)-dependent, if there exists a sequence $\theta = \{\theta_r\}_{r \in \mathbb{R}_0^+}$, $\theta_r \downarrow 0$ as $r \to \infty$ such that for any finite disjoint sets $I, J \subset \mathbb{Z}^d$ with $\operatorname{dist}(I, J) = r$, and any functions $f \in BL(|I|)$, $g \in BL(|J|)$, one has

$$|\operatorname{cov}(f(X_I), g(X_J))| \leq \min\{|I|, |J|\} \operatorname{Lip}(f) \operatorname{Lip}(g) \theta_r.$$

Possible choice of θ_r:

$$\theta_r = \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d : \|j - i\|_\infty \geq r} |\operatorname{cov}(X(i), X(j))|.$$

CLT for (BL, θ)-dependent stationary random fields

Theorem (Bulinski & Shashkin, 2007)

Let $Z = \{Z(j), j \in \mathbb{Z}^d\}$ be a (BL, θ)-dependent strictly stationary centered square-integrable random field. Then, for any sequence of regularly growing sets $U_n \subset \mathbb{Z}^d$, one has

$$\sum_{j \in U_n} Z(j)/\sqrt{|U_n|} \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

as $n \to \infty$, with

$$\sigma^2 = \sum_{j \in \mathbb{Z}^d} \operatorname{cov}\left(Z\left(0\right), Z\left(j\right)\right).$$

Special case - Shot noise random fields

The above CLT holds for a stationary shot noise random field $X = \{X(t), t \in \mathbb{R}^d\}$ given by $X(t) = \sum_{i\in\mathbb{N}} \xi_i \varphi(t - x_i)$ where

- $\{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$
- $\{\xi_i\}$ is a family of i.i.d. non–negative random variables with $\mathrm{E}\,\xi_i^2 < \infty$ and the characteristic function φ_ξ
- $\{\xi_i\}$, $\{x_i\}$ are independent
- $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ is a bounded and uniformly continuous Borel function with

$$\varphi(t) \le \varphi_0(\|t\|_2) = O\left(\|t\|_2^{-\alpha}\right) \quad \text{as} \quad \|t\|_2 \to \infty$$

for a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha > 3d$, and

$$\int_{\mathbb{R}^d} \left| \exp\left\{ \lambda \int_{\mathbb{R}^d} \left(\varphi_\xi(s\varphi(t)) - 1\right) dt \right\} \right| ds < \infty.$$

Special case - Gaussian random fields

Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with the following properties:

- $X(0) \sim \mathcal{N}\left(a, \tau^2\right)$
- has a continuous covariance function $r(\cdot)$
- $\exists\, \alpha > d : |r(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ as $\|t\|_2 \to \infty$

Special case - Gaussian random fields

Let X be the above Gaussian random field and $u \in \mathbb{R}$. Then,

$$\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \, e^{-\frac{(u-a)^2}{\tau^2(1+r)}} \, dr \, dt,$$

where $\rho(t) = \mathrm{corr}(X(0), X(t))$. In particular, for $u = a$

$$\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin\left(\rho(t)\right) \, dt.$$

Positively or negatively associated random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- stochastically continuous (evtl. not second order!)
- $\sigma^2(u) \in (0, \infty)$
- $\mathrm{P}(X(0) = u) = 0$ for the chosen level $u \in \mathbb{R}$
- positively (**PA**) or negatively (**NA**) associated.

Then the above CLT holds (Karcher (2012)).

Association

A random field $X = \left\{X(t), t \in \mathbb{R}^d\right\}$ is called positively (**PA**) or negatively (**NA**) associated if

$$\operatorname{cov}\left(f\left(X_I\right), g\left(X_J\right)\right) \geq 0 \quad (\leq 0, \text{ resp.})$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any bounded coordinatewise non–decreasing functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$.

Special cases

Subclasses of **PA** or **NA**

- infinitely divisible
- max-stable
- α-stable

random fields

Special cases: Max-stable random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary max-stable random field with spectral representation

$$X(t) = \max_{i \in \mathbb{N}} \xi_i f_t(y_i), \quad t \in \mathbb{R}^d,$$

where $f_t : E \to \mathbb{R}_+$ is a measurable function defined on the measurable space (E, μ) for all $t \in \mathbb{R}^d$ with

$$\int_E f_t(y)\, \mu(dy) = 1, \quad t \in \mathbb{R}^d,$$

and $\{(\xi_i, y_i)\}_{i \in \mathbb{N}}$ is the Poisson point process on $(0, \infty) \times E$ with intensity measure $\xi^{-2} d\xi \times \mu(dy)$. Assume that

$$\int_{\mathbb{R}^d} \int_E \min\{f_0(y), f_t(y)\}\, \mu(dy)\, dt < \infty$$

and $\|f_s - f_t\|_{L^1} \to 0$ as $s \to t$.

Special cases: α-stable random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary α-stable random field ($\alpha \in (0,2)$, for simplicity $\alpha \neq 1$) with spectral representation

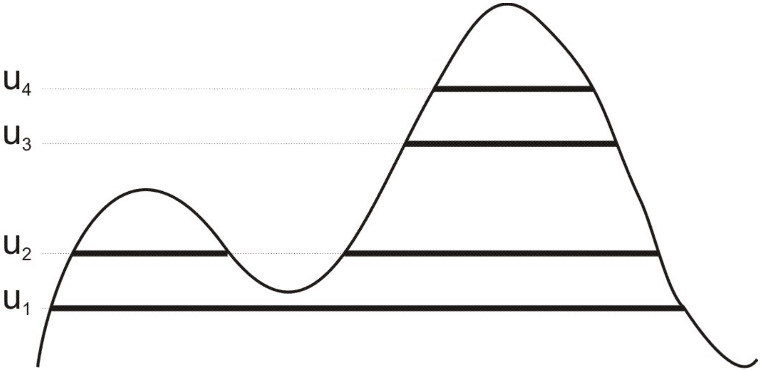
$$X(t) = \int_E f_t(x)\,\Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered α–stable random measure on space E with control measure m and skewness intensity $\beta : E \to [-1,1]$, $f_t : E \to \mathbb{R}_+$ is a measurable function on (E, m) for all $t \in \mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} \left(\int_E \min\{|f_0(x)|^\alpha, |f_t(x)|^\alpha\}\, m(dx) \right)^{1/(1+\alpha)} dt < \infty$$

and $\int_E |f_s(x) - f_t(x)|^\alpha\, m(dx) \to 0$ as $s \to t$.

Multi-dimensional CLT



$$S_{\vec{u}}(W_n) = \left(|A_{u_1}(X, W_n)|, \dots, |A_{u_r}(X, W_n)|\right)^\top$$

$$\Psi(\vec{u}) = \left(\Psi((u_1 - a)/\tau), \dots, \Psi((u_r - a)/\tau)\right)^\top$$

Theorem (Multi-dimensional CLT)

Let X be the above Gaussian random field and $u_k \in \mathbb{R}$, $k = 1, \ldots, r$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$|W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n| \right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\vec{u}))$$

as $n \to \infty$. Here, $\Sigma(\vec{u}) = (\sigma_{lm}(\vec{u}))_{l,m=1}^{r}$ with

$$\sigma_{lm}(\vec{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp\left\{ -\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-r^2)} \right\} dr\, dt.$$

Theorem (Statistical version of the CLT)

Let X be the above Gaussian random field, $u_k \in \mathbb{R}$, $k = 1, \dots, r$ and $(W_n)_{n \in \mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^{r}$ be statistical estimates for the nondegenerate asymptotic covariance matrix $\Sigma(\vec{u})$, such that for any $l, m = 1, \dots, r$

$$\hat{c}_{nlm} \xrightarrow{p} \sigma_{lm}(\vec{u}) \text{ as } n \to \infty.$$

Then

$$\hat{C}_n^{-1/2} \, |W_n|^{-1/2} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u}) \, |W_n| \right) \xrightarrow{d} \mathcal{N}(0, I).$$

Hypothesis testing

$$H_0 : X \text{ Gaussian vs. } H_1 : X \text{ Non-Gaussian}$$

Test statistic:

$$T = |W_n|^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\,|W_n|\right)^\top \hat{C}_n^{-1} \left(S_{\vec{u}}(W_n) - \Psi(\vec{u})\,|W_n|\right)$$

We know $T \xrightarrow{d} \chi^2_r$. Reject null-hypothesis if $T > \chi^2_{r,1-\nu}$.

Numerical results

Series	FTR6.3	FTR6.6	Sim. Gaussian
Resolution	218*x*138	218*x*138	218*x*138
Realizations	100	100	100
1 level			
Rejected fields ($\nu = 1\%$)	0	0	1
3 levels			
Rejected fields ($\nu = 1\%$)	5	9	3
5 levels			
Rejected fields ($\nu = 1\%$)	20	21	3
7 levels			
Rejected fields ($\nu = 1\%$)	34	31	5
9 levels			
Rejected fields ($\nu = 1\%$)	62	60	5

Further results

- CLT for the volume as level $u \to \infty$
  - Isotropic Gaussian random fields: Ivanov & Leonenko (1989)
  - **PA**-random fields: Demichev & Olszewski (2015)
- FCLT (variable $u \in \mathbb{R}$):
  - Volume for second order **A**-random fields with a.s. continuous paths and bounded density in Skorokhod space: Meschenmoser & Shashkin (2011)
  - Volume for random fields with a.s. continuous paths and bounded density in Skorokhod space (evtl. not second order!): Karcher (2012)

FCLT for excursion sets (Meschenmoser & Shashkin, 2011)

Let $\{X(t)\}_{t\in\mathbb{R}^d}$ be a stationary, measurable, associated random field fulfilling some integrability and regularity assumptions.
Let $\{W_n\}_{n\in\mathbb{N}}$ be a VH-growing sequence of compact sets of $\mathbb{R}^d$.
Then the sequence of stochastic processes defined by

$$Y_n(u) := \frac{\int_{W_n} \mathbf{1}_{[u,\infty)}(X(t))\,dt - |W_n| \cdot \mathbb{E}[\mathbf{1}_{[u,\infty)}(X(0))]}{\sqrt{|W_n|}}$$

converges in distribution to a centered Gaussian process Y with covariance function

$$\operatorname{Cov}(Y(u_1), Y(u_2)) = \int_{\mathbb{R}^d} \mathbb{P}(X(0) > u_1, X(t) > u_2) - \mathbb{P}(X(0) > u_1) \cdot \mathbb{P}(X(t) > u_2)\,dt$$

as $n \to \infty$ in the Skorokhod topology.

Replace indicator functions by more general functions?

Consider the space V of Lipschitz continuous functions with norm

$$\|f\|_{\mathrm{Lip}} := \mathrm{Lip}\, f + |f(0)|.$$

Theorem (Kampf & S. (2015))

Let $\{X(t)\}_{t\in\mathbb{R}^d}$ be a stationary and measurable random field. Assume there $n \in \mathbb{N}$, $\delta > 4$ and $C, l > 0$ with $n/d > \max\{l + \delta/(\delta - 2), \delta/(\delta - 4)\}$ such that

$$\alpha_\gamma(r) \le Cr^{-n}\gamma^l \text{ for all } \gamma \ge 2\kappa_d,\ r > 0,$$

and

$$\mathbb{E}X(0)^\delta < \infty.$$

Let $\{W_n\}_{n\in\mathbb{N}}$ be a VH-growing sequence of compact sets of $\mathbb{R}^d$. Then the sequence of stochastic processes defined by

$$\Phi_n(f) := \frac{\int_{W_n} f(X(t))\,dt - |W_n| \cdot \mathbb{E}[f(X(0))]}{\sqrt{|W_n|}},\ f \in V,$$

converges in distribution to a centered Gaussian process Φ with covariance function

$$\mathrm{Cov}(\Phi(f), \Phi(g)) = \int_{\mathbb{R}^d} \mathrm{Cov}\left(f(X(0)), g(X(t))\right) dt$$

as $n \to \infty$ in the weak topology.

Sketch of the proof:

According to Oppel (1973) it suffices to show:

1. The finite-dimensional distributions converge appropriately.
2. The processes Φ_n have linear and continuous paths.
3. The process Φ exists and has a version with linear and continuous paths.

How we show it?

1. Employ Cramér-Wold-technique
2. Trivial

Show 3. (Continuity of limiting process):
Employ theory of GB- and GC-sets

What is this theory about?
For a Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$ the isonormal process is the centered Gaussian process Φ with

$$\mathrm{Cov}(\Phi(f), \Phi(g)) = \langle f, g \rangle.$$

Sets, on which a version of the isonormal process has bounded / continuous paths, are called GB / GC -sets.

How to apply this theory to our problem?

Define scalar product such that Φ becomes the isonormal process:

$$\begin{aligned}\langle f, g\rangle &= \mathrm{Cov}(\Phi(f), \Phi(g)) \\ &= \int_{\mathbb{R}^d} \mathrm{Cov}\left(f(X(0)), g(X(t))\right) dt\end{aligned}$$

This is a symmetric, non-negative definite, bilinear form. Passing to equivalence classes and completion we obtain a Hilbert space.

We have to show that (the projection of)

$$B := \{f \in V \mid \|f\|_{\mathrm{Lip}} \leq 1\}$$

is a GB-set.

Let $N(\epsilon)$ be the minimal number of elements of an ϵ-net on B. An ϵ-net on B are elements $f_1, \ldots, f_n \in B$ such that

$$\forall_{g \in B} : \exists_{i \in \{1,\ldots,n\}} : \|f_i - g\|_{\langle \cdot, \cdot \rangle} < \epsilon.$$

It is well known (see e.g. Dudley, 1999) that it suffices to show

$$\int_0^1 (\log N(\epsilon))^{1/2} \, d\epsilon < \infty. \tag{1}$$

For $m \in \mathbb{N}$ and $c > 0$ consider the Lipschitz functions f with

- $f(0) = 0$
- On each interval $[(k-1)c, kc]$, $k = -m+1, \ldots, m$, the function f is either
  increasing with constant slope 1 or
  decreasing with constant slope -1.
- On $(-\infty, -mc]$ and $[mc, \infty)$, the function f is constant.

Under appropriate inequalities on ϵ, c and m, these functions form an ϵ-net with (1).

Does the asymptotic variance $\mathrm{Var}(\Phi(f)), f \in V$, vanish?

For Gaussian random fields with non-negative covariance function:

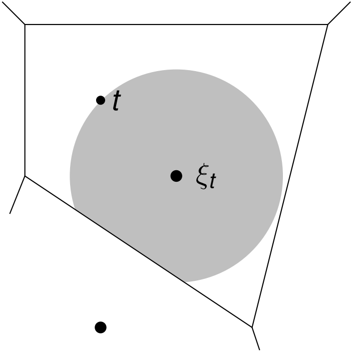
$$\mathrm{Var}(\Phi(f)) = 0 \iff \mathrm{Var}(X(0)) = 0$$
$$\text{or } f \text{ is constant}$$

Construction of a field $X(t)_{t\in\mathbb{R}^d}$ that fulfills all assumptions of our theorem, but $\mathrm{Var}(\Phi(f)) = 0$ for all $f \in V$:

Consider a Poisson-Voronoi-mosaic, i.e. a random partition of $\mathbb{R}^d$ into convex polytopes.

$$X(t) := |\{v \in C(t) \mid \|v - \xi_t\| \le \|t - \xi_t\|\}| / |C(t)|, \quad t \in \mathbb{R}^d,$$

where $C(t)$ is the cell in which t lies and ξ_t is its nucleus.



$$\int_C f(X(t))\,dt = \int_0^1 f(x)\,dx \cdot |C|$$

for each (random) cell C.

$$\int_W f(X(t))\,dt \approx \int_0^1 f(x)\,dx \cdot |W|$$

for each (large, deterministic) compact set $W \subseteq \mathbb{R}^d$, strengthened by mixing properties of Poisson-Voronoi-mosaic.

$\Rightarrow$ The variance vanishes asymptotically.

Further issue: Orthogonality in $(V, \langle \cdot, \cdot \rangle)$

For a certain class of random fields constructed with help of Lévy-Meixner-systems we obtained explicit ONBs.

Open problems

- Proof of the last theorem (FCLT) for a space larger than V and under association instead of mixing
- LTs for stationary long range dependent random fields

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