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Scaling transition for nonlinear random fields with long-range dependence

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Joint work with Vytautė Pilipauskaitė (Nantes/Vilnius)

Outline:

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1. Scaling limit: ‘a summary of dependence structure’

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2. Scaling transition for Gaussian LRD RFs

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- Scaling (partial sums) limits of any *weakly dependent* 2nd order process X coincide with Brownian motion (Donsker's theorem)
- Scaling limit of a stationary process X is self-similar (Lamperti, 1962) and provides a 'large-scale summary of dependence structure of X'

Anisotropic scaling limit: as $\lambda \to \infty$

$$A_{\lambda,\gamma}^{-1} \sum_{(t,s)\in K_{[\lambda x, \lambda^\gamma y]}} X(t,s) \quad \xrightarrow{\mathrm{fdd}} \quad V_\gamma^X(x,y), \quad (x,y) \in \mathbb{R}_+^2. \qquad (1)$$

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- limit RF V_γ^X depends on γ (also on the law of X)

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- Does and how $V^X = \{V_\gamma^X, \gamma > 0\}$ reflect the dependence in X along different directions?

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$$\lambda^{H(\gamma)} V(x, y) \stackrel{\text{fdd}}{=} V(\lambda x, \lambda^\gamma y) \quad \forall\, \lambda > 0. \tag{2}$$

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- 'Nontrivial' scaling diagram is intrinsically related to *long-range dependence* (LRD): $\sum_{(t,s)\in\mathbb{Z}^2} |\mathrm{cov}(X(0,0), X(t,s))| = \infty$

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- Aggregation of teletraffic and macroeconomic models

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  Anisotropic quadratic variations:

$$QV^X_{n,m} \quad := \sum_{(i/n, j/m) \in [0,1]^2} |\Delta_{1/n,1/m} X(i/n, j/m)|^2,$$

$\Delta_{1/n,1/m} X(t,s) := X(t+1/n, s+1/m) - X(t, s+1/m) - X(t+1/n, s) + X(t,s)$ is the double difference

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$$\lambda^{H(\gamma)}(X(t+\lambda x, s+\lambda^\gamma y) - X(t+\lambda x, s) - X(t, s+\lambda^\gamma y) + X(t,s))$$
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$$X(t,s) = \int_{\mathbb{R}^2} (g(t-u, s-v) - \tilde{g}_1(-u, s-v) - \tilde{g}_2(t-u, -v) + \tilde{g}_{12}(-u,-v)) L(\mathrm{d}u, \mathrm{d}v),$$

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- $V_\pm^X$ called the *unbalanced scaling limits of* X

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&V_\gamma^X \overset{\mathrm{fdd}}{=} V_+^X \quad (\forall\, \gamma > \gamma_0), \qquad V_\gamma^X \overset{\mathrm{fdd}}{=} V_-^X \quad (\forall\, \gamma < \gamma_0),\\
&V_+^X \overset{\mathrm{fdd}}{\neq} a V_-^X \quad (\forall\, a > 0).
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- $V_\pm^X$ called the *unbalanced scaling limits of* X
- $V_{\gamma_0}^X$ called the *well-balanced scaling limit of* X

2. Scaling transition for Gaussian LRD RFs

$X = \{X(t,s); (t,s) \in \mathbb{Z}^2\}$: a stationary random field (RF) on $\mathbb{Z}^2$ s.t. scaling limits $V_\gamma^X = \{V_\gamma^X(x,y); (x,y) \in \mathbb{R}_+^2\}$ (1) exist for any $\gamma > 0$

Def We say that X *exhibits scaling transition* if $\exists \gamma_0 > 0$ s.t.

$$
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Scaling transition for Gaussian LRD RFs on $\mathbb{Z}^2$

A zero mean stationary Gaussian RF $X = \{X(t,s); (t,s) \in \mathbb{Z}^2\}$ is completely described by spectral density $f = f(x,y) \ge 0$, $(x,y) \in [-\pi,\pi]^2$

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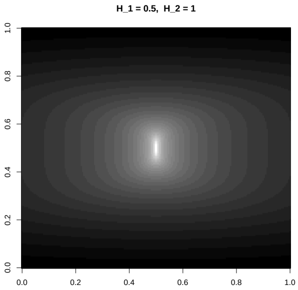
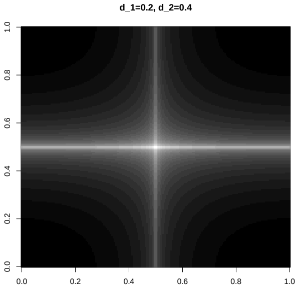
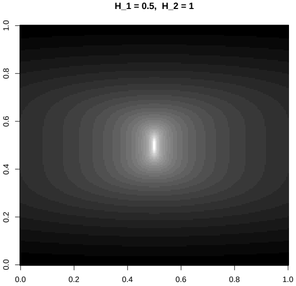
- g: bdd& ctn, $g(0,0) > 0$



Type I sp. density f_{I}, $H_1 = 0.5$, $H_2 = 1$



Type II sp. density f_{II}, $d_1 = 0.2$, $d_2 = 0.4$



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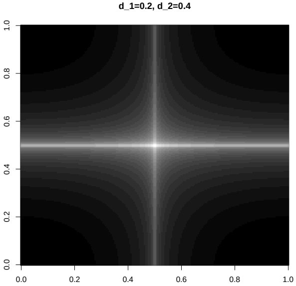

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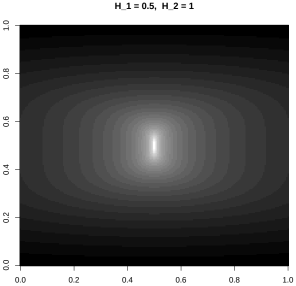

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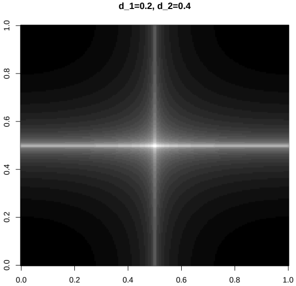

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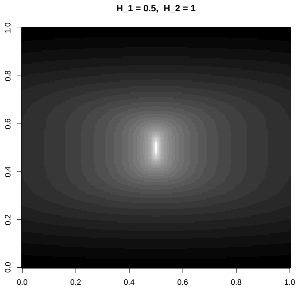
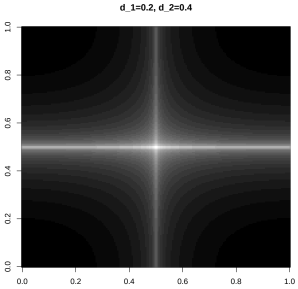


Type I sp. density f_{I}, $H_1 = 0.5$, $H_2 = 1$



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- f_{II} include fractionally integrated class $|1 - \mathrm{e}^{-\mathrm{i}x}|^{-2d_1}|1 - \mathrm{e}^{-\mathrm{i}y}|^{-2d_2}$

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(ii) Let X be a stationary zero mean Gaussian RF on $\mathbb{Z}^2$ with Type II spectral density in (3), $0 < d_1, d_2 < 1/2$. Then X *does not exhibit scaling transition*.

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$$Y(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a(t-u, s-v)\varepsilon(u,v), \qquad (t,s)\in\mathbb{Z}^2, \tag{4}$$

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- (6) implies $\sum_{(t,s)\in\mathbb{Z}^2} a(t,s)^2 < \infty$, $\sum_{(t,s)\in\mathbb{Z}^2} |a(t,s)| = \infty$, i.e. Y in (4)-(5) is a well-defined LRD RF

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- $\Delta Y(t,s) := (1/4)\sum_{|u|+|v|=1}(Y(t+u,s+v) - Y(t,s))$: (discrete) Laplace operator (elliptic)
- $0 < d < 1/2$: order of fractional integration
- $(1-z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j, \psi_j(d) := \Gamma(j-d)/\Gamma(j+1)\Gamma(-d)$
- $\{\varepsilon(t,s), (t,s) \in \mathbb{Z}^2\}$: standardized i.i.d.

- Stationary solution of (7): zero-mean finite variance RF:

$$Y(t,s) = (-\Delta)^{-d}\varepsilon(t,s) \;=\; \sum_{(u,v)\in\mathbb{Z}^2} a(u,v)\varepsilon(t-u,s-v),$$

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- Y satisfies Assumption (A2) with
$q_1 = q_2 = 2(1-d)\in(1,2)$, $Q = 1/(1-d)\in(1,2)$ and a constant angular function $L_0(z) = A$, $z\in[-1,1]$.

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- MA coefficients satisfy Assumption (A2) with $q_1 = 3/2 - d$, $q_2 = 2q_1$ and a continuous angular function $L_0(z), z \in [-1,1]$ given by

$$L_0(z) \;=\; \begin{cases} \frac{z^{d-3/2}}{\Gamma(d)\sqrt{2\pi(1-\theta)}} \exp\Big\{-\frac{\sqrt{(1/z)^2-1}}{2(1-\theta)}\Big\}, & 0 < z \le 1, \\ 0, & -1 \le z \le 0. \end{cases}$$

Thm 2 Let Y be a linear RF in (4)-(5) satisfying Assumptions (A1)-(A2), $\frac{1}{2q_1} + \frac{1}{q_2} \neq 1, \frac{1}{q_1} + \frac{1}{2q_2} \neq 1$.

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- Thm 2 is similar to Thm 1
- There is a 'heuristic' 1-1 correspondence between parameters H_1, H_2 in Thm 1 and q_1, q_2 in Thm 2:

$$H_i = 2q_i\Big(\frac{1}{q_1} + \frac{1}{q_2} - 1\Big), \qquad q_i = H_i\Big(\frac{1}{H_1} + \frac{1}{H_2} - \frac{1}{2}\Big), \quad i = 1, 2.$$

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Question: what happens if RF X is nonlinear?

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Assumption (A3)$_k$ For $k \in \mathbb{N}_+$, $\mathrm{E}|\varepsilon|^{2k} < \infty$ and

$$X(t,s) := A_k(Y(t,s)), \qquad (t,s) \in \mathbb{Z}^2 \tag{9}$$

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Assumption (A4)$_k$ $\varepsilon(0,0) \overset{\mathrm{d}}{=} Z$ and $Y(0,0) \overset{\mathrm{d}}{=} Z$ have standard normal distribution $Z \sim N(0,1)$ and

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Central and noncentral limit theorems for nonlinear functionals (Gaussian and polynomial chaos):
Dobrushin and Major (1979), Taqqu (1975, 1979), S. (1982), Breuer and Major (1983), Giraitis and S. (1985), Avram and Taqqu (1987), Ho and Hsing (1997), Leonenko (1999), Arcones (2000), Nualart and Peccati (2005), Bai and Taqqu (2014) + many more

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Prop 1 Let RF $X = A_k(Y)$ satisfy assumptions (A1), (A2) and (A3)$_k$.

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(R5) In the case of Gaussian underlying RF Y in (4), the above conclusions hold for $X = G(Y)$ and a general nonlinear function G with k equal to the Hermite rank of G

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- For any $h \in L^2(\mathbb{R}^{2k})$ the k-tuple Itô-Wiener integral

  $$\int_{\mathbb{R}^{2k}} h((u,v)_k) \mathrm{d}^k W = \int_{\mathbb{R}^{2k}} h(u_1, v_1, \cdots, u_k, v_k)\, W(\mathrm{d}u_1, \mathrm{d}v_1) \cdots W(\mathrm{d}u_k, \mathrm{d}v_k)$$

  is well-defined and satisfies

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$$\lambda^{-H(\gamma)} S_{\lambda,\gamma}^X(x,y) \ \xrightarrow{\mathrm{fdd}} \ xZ_k^+(y). \tag{14}$$

(iii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and $1 \le k < 1/p_1$.

Thm 4 (i) Z_k^+ and Z_k^- are well-defined for $1 \le k < 1/p_2$ and $1 \le k < 1/p_1$, respectively, as Itô-Wiener stochastic integrals. They have zero mean, finite variance, stationary increments and are self-similar with respective indices $H_{2k}^+ := 1 - kp_2/2 \in (1/2, 1)$ and $H_{1k}^- := 1 - kp_1/2 \in (1/2, 1)$.

(ii) Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and $1 \le k < 1/p_2$. Then for any $\gamma > \gamma_0$

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where $H(\gamma) := \gamma + H_{1k}^-$ and $c(\gamma) := \|h_-(1;\cdot)\|_k^2 > 0$.

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- Similarly as in linear case $k = 1$ ($X = A_1(Y) = Y$) unbalanced scaling limits of $X = A_k(Y)$ for $1 \le k < P$ have special dependence structure: either independent, or completely dependent increments along one of the the coordinate axes

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- The point $kp_2 = 1$ at which scaling limit of $X = A_k(Y)$ for $\gamma > \gamma_0$ changes from 'Hermite slide' $xZ_k^+(y)$ to FBSheet $B_{H_{1k}^+, 1/2}(x, y)$ coincides with the point where the covariance function of $X = A_k(Y)$ changes from vertical LRD to vertical SRD:

$$\sum_{s \in \mathbb{Z}} |r_X(0, s)| \begin{cases} = \infty, & kp_2 \le 1, \\ < \infty, & kp_2 > 1. \end{cases}$$

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$$\sum_{s\in\mathbb{Z}} |r_X(0,s)| \begin{cases} = \infty, & kp_2 \le 1, \\ < \infty, & kp_2 > 1. \end{cases}$$

- The point $kp_1 = 1$ at which scaling limit of $X = A_k(Y)$ for $\gamma < \gamma_0$ changes from 'Hermite slide' $yZ^-_k(x)$ to FBSheet $B_{1/2, H^-_{2k}}(x,y)$ coincides with the point where the covariance function of $X = A_k(Y)$ changes from horizontal LRD to horizontal SRD:

$$\sum_{t\in\mathbb{Z}} |r_X(t,0)| \begin{cases} = \infty, & kp_1 \le 1, \\ < \infty, & kp_1 > 1. \end{cases}$$

Thm 6 Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and

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$$\mathrm{Var}(S^X_{\lambda,\gamma}) \ \sim \ \sigma^2_X \lambda^{1+\gamma},$$

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Thm 7 Let $X = G(Y)$ satisfy Assumption (A4)$_k$.

Thm 6 Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and

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$$\mathrm{Var}(S^X_{\lambda,\gamma}) \sim \sigma_X^2 \lambda^{1+\gamma},$$

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Thm 7 Let $X = G(Y)$ satisfy Assumption (A4)$_k$. Assume w.l.g. that G has Hermite expansion $G(x) = H_k(x) + \sum_{j=k+1}^{\infty} c_j H_j(x)/j!$.

Thm 6 Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and (A3)$_k$ and

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(i) Let $1 \le k < P$. Then RF X satisfies all statements of Thms 3-5.

Thm 6 Let RFs Y and $X = A_k(Y)$ satisfy Assumptions (A1), (A2) and $(\mathrm{A3})_k$ and

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Then for any $\gamma > 0$

$$\mathrm{Var}(S^X_{\lambda,\gamma}) \;\sim\; \sigma^2_X \lambda^{1+\gamma},$$

where $\sigma^2_X := \sum_{(t,s)\in\mathbb{Z}^2} \mathrm{Cov}(X(0,0), X(t,s)) \in (0,\infty)$. Moreover,

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(i) Let $1 \le k < P$. Then RF X satisfies all statements of Thms 3-5.

(ii) Let $k > P$. Then RF X satisfies the statements of Thm 6.

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