Selfdecomposable distributions in free probability

15. august 2016

Steen Thorbjørnsen, Dept. of Mathematics, Univ. of Aarhus

Selfdecomposable distributions in free probability

15. august 2016

Steen Thorbjørnsen, Dept. of Mathematics, Univ. of Aarhus

Takahiro Hasebe, Dept. of Mathematics, Hokkaido University, Japan.

Noriyoshi Sakuma, Dept. of Mathematics, Aichi Univ. of Educ., Japan.

Free independence

Recall that two random variables X and Y are independent, if

$$\mathbb{E}\big\{\big(f(X)-\mathbb{E}\{f(X)\}\big)\big(g(Y)-\mathbb{E}\{f(Y)\}\big)\big\}=0,$$

for any bounded Borel functions $f, g\colon \mathbb{R}\to\mathbb{R}$.

Free independence

Recall that two random variables X and Y are independent, if

$$\mathbb{E}\big\{\big(f(X)-\mathbb{E}\{f(X)\}\big)\big(g(Y)-\mathbb{E}\{f(Y)\}\big)\big\}=0,$$

for any bounded Borel functions $f,g\colon\mathbb{R}\to\mathbb{R}$.

Two random variables a and b are called freely independent, if they satisfy the condition:

$$\mathbb{E}\big\{\big[f_1(a)-\mathbb{E}\{f_1(a)\}\big]\big[f_2(b)-\mathbb{E}\{f_2(b)\}\big]\cdots\big[f_k(a)-\mathbb{E}\{f_k(a)\}\big]\big\}=0,$$

for any bounded Borel-functions $f_1,f_2,\dots,f_k$.

Free independence

Recall that two random variables X and Y are independent, if

$$\mathbb{E}\big\{\big(f(X) - \mathbb{E}\{f(X)\}\big)\big(g(Y) - \mathbb{E}\{f(Y)\}\big)\big\} = 0,$$

for any bounded Borel functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

Two random variables a and b are called freely independent, if they satisfy the condition:

$$\mathbb{E}\big\{\big[f_1(a) - \mathbb{E}\{f_1(a)\}\big]\big[f_2(b) - \mathbb{E}\{f_2(b)\}\big]\cdots\big[f_k(a) - \mathbb{E}\{f_k(a)\}\big]\big\} = 0,$$

for any bounded Borel-functions $f_1, f_2, \ldots, f_k$.

Except for trivial cases, free independence entails that

$$ab \neq ba.$$

Non-commutative random variables

Think of a and b as Hermitian operators on an infinite dimensional Hilbert space $\mathcal{H}$,

Non-commutative random variables

Think of a and b as Hermitian operators on an infinite dimensional Hilbert space $\mathcal{H}$, and

$$\mathbb{E}\{a\} = \langle a\xi, \xi\rangle$$

for some unit vector ξ from $\mathcal{H}$.

Non-commutative random variables

Think of a and b as Hermitian operators on an infinite dimensional Hilbert space $\mathcal{H}$, and

$$\mathbb{E}\{a\} = \langle a\xi, \xi\rangle$$

for some unit vector ξ from $\mathcal{H}$.

Then $f(a)$ is defined in terms of spectral theory (for any bounded Borel-function f).

Non-commutative random variables

Think of a and b as Hermitian operators on an infinite dimensional Hilbert space $\mathcal{H}$, and

$$\mathbb{E}\{a\} = \langle a\xi, \xi\rangle$$

for some unit vector ξ from $\mathcal{H}$.

Then $f(a)$ is defined in terms of spectral theory (for any bounded Borel-function f).

Moreover, there exists a unique probability measure μ_a on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} f(t)\,\mu_a(\mathrm{d}t) = \langle f(a)\xi, \xi\rangle$$

for any bounded Borel-function f.

Non-commutative random variables

Think of a and b as Hermitian operators on an infinite dimensional Hilbert space $\mathcal{H}$, and

$$\mathbb{E}\{a\} = \langle a\xi, \xi\rangle$$

for some unit vector ξ from $\mathcal{H}$.

Then $f(a)$ is defined in terms of spectral theory (for any bounded Borel-function f).

Moreover, there exists a unique probability measure μ_a on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} f(t)\,\mu_a(\mathrm{d}t) = \langle f(a)\xi, \xi\rangle$$

for any bounded Borel-function f.

The measure μ_a is called the (spectral) distribution of a.

Free convolution

Let μ and ν be probability measures on $\mathbb{R}$, and consider *freely independent* Hermitian operators a and b, such that $a \sim \mu$ and $b \sim \nu$.

Free convolution

Let μ and ν be probability measures on $\mathbb{R}$, and consider *freely independent* Hermitian operators a and b, such that $a \sim \mu$ and $b \sim \nu$.

Then the free convolution $\mu \boxplus \nu$ is defined by:

$$a + b \sim \mu \boxplus \nu.$$

Free infinite divisibility

By $\mathcal{ID}(\boxplus)$ we denote the class of ⊞-infinitely divisible probability measures on $\mathbb{R}$, i.e.

$$\mu \in \mathcal{ID}(\boxplus) \iff \forall n \in \mathbb{N}\ \exists \mu_n \in \mathcal{P}(\mathbb{R})\colon\ \mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

The free cumulant transform

Let μ be a probability measure on $\mathbb{R}$, and consider its Cauchy (or Stieltjes) transform:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

The free cumulant transform

Let μ be a probability measure on $\mathbb{R}$, and consider its Cauchy (or Stieltjes) transform:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

The free analog of $\log \hat{\mu}$ is the free cumulant transform:

$$\mathcal{C}_\mu(z) = zG_\mu^{\langle -1\rangle}(z) - 1, \qquad (z \in \mathcal{D} \subseteq \mathbb{C}^-).$$

The free cumulant transform

Let μ be a probability measure on $\mathbb{R}$, and consider its Cauchy (or Stieltjes) transform:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

The free analog of $\log \hat{\mu}$ is the free cumulant transform:

$$\mathcal{C}_\mu(z) = zG_\mu^{\langle -1 \rangle}(z) - 1, \qquad (z \in \mathcal{D} \subseteq \mathbb{C}^-).$$

Theorem [Voiculescu, Maasen, Bercovici-Voiculescu]. For any probability measures μ_1, μ_2 on $\mathbb{R}$ we have that

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z).$$

The Free Lévy-Khintchine-representation

Theorem [Bercovici & Voiculescu]. Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform $\mathcal{C}_\mu$.

The Free Lévy-Khintchine-representation

Theorem [Bercovici & Voiculescu]. Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform $\mathcal{C}_\mu$.

Then μ is ⊞-infinitely divisible, if and only if $\mathcal{C}_\mu$ has a representation in the form:

$$\mathcal{C}_\mu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz 1_{[-1,1]}(t) \Big)\, \rho(\mathrm{d}t), \quad (z \in \mathbb{C}^-),$$

The Free Lévy-Khintchine-representation

Theorem [Bercovici & Voiculescu]. Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform $\mathcal{C}_\mu$.

Then μ is ⊞-infinitely divisible, if and only if $\mathcal{C}_\mu$ has a representation in the form:

$$\mathcal{C}_\mu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz 1_{[-1,1]}(t) \Big)\, \rho(\mathrm{d}t), \quad (z \in \mathbb{C}^-),$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$.

The Free Lévy-Khintchine-representation

Theorem [Bercovici & Voiculescu]. Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform $\mathcal{C}_\mu$.

Then μ is ⊞-infinitely divisible, if and only if $\mathcal{C}_\mu$ has a representation in the form:

$$\mathcal{C}_\mu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz 1_{[-1,1]}(t) \Big) \, \rho(\mathrm{d}t), \quad (z \in \mathbb{C}^-),$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$.

In that case, the *free characteristic triplet* (a, ρ, η) is uniquely determined.

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$\mathcal{ID}(*) \ni \mu$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \frac{au^2}{2} + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut 1_{[-1,1]}(t)\big)\rho(\mathrm{d}t)$$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \frac{au^2}{2} + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \tfrac{au^2}{2} + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

$$\longleftrightarrow \qquad \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\rho(\mathrm{d}t)$$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \tfrac{au^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\right)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

$$\longleftrightarrow \mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz1_{[-1,1]}(t)\Big)\rho(\mathrm{d}t)$$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log\hat{\mu}(u) = \mathrm{i}\eta u - \tfrac{au^2}{2} + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

$$\longleftrightarrow \mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow \Lambda(\mu) := \nu \in \mathcal{ID}(\boxplus).$$

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \frac{au^2}{2} + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

$$\longleftrightarrow \mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz1_{[-1,1]}(t)\Big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow \Lambda(\mu) := \nu \in \mathcal{ID}(\boxplus).$$

Key Properties:

- $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$ for any μ_1, μ_2 in $\mathcal{ID}(*)$.

The Bercovici-Pata bijection

Definition. The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows:

$$\mathcal{ID}(*) \ni \mu \longleftrightarrow \log \hat{\mu}(u) = \mathrm{i}\eta u - \frac{au^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\right)\rho(\mathrm{d}t)$$

$$\longleftrightarrow (a, \rho, \eta)$$

$$\longleftrightarrow \mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\rho(\mathrm{d}t)$$

$$\longleftrightarrow \Lambda(\mu) := \nu \in \mathcal{ID}(\boxplus).$$

Key Properties:

- $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$ for any μ_1, μ_2 in $\mathcal{ID}(*)$.
- $\Lambda(D_c\mu) = D_c\Lambda(\mu)$ for any μ in $\mathcal{ID}(*)$ and c in $\mathbb{R}$.

Free Selfdecomposability

A measure μ on $\mathbb{R}$ is ⊞-selfdecomposable, if

$$\forall c \in (0,1)\ \exists \mu_c \in \mathcal{P}(\mathbb{R})\colon\ \mu = D_c\mu \boxplus \mu_c.$$

Free Selfdecomposability

A measure μ on $\mathbb{R}$ is ⊞-selfdecomposable, if

$$\forall c \in (0,1)\ \exists \mu_c \in \mathcal{P}(\mathbb{R})\colon\ \mu = D_c\mu \boxplus \mu_c.$$

In this case μ and μ_c are necessarily ⊞-infinitely divisible.

Λ preserves selfdecomposability

Theorem [Barndorff-Nielsen+T]. For a $*$-infinitely divisible probability measure μ, we have that

$$\mu \text{ is } *\text{-sd} \iff \Lambda(\mu) \text{ is } \boxplus\text{-sd}.$$

Λ preserves selfdecomposability

Theorem [Barndorff-Nielsen+T]. For a $*$-infinitely divisible probability measure μ, we have that

$$\mu \text{ is } *\text{-sd} \iff \Lambda(\mu) \text{ is } \boxplus\text{-sd}.$$

Corollary. A probability measure ν on $\mathbb{R}$ is $\boxplus$-s.d., if and only if $\nu \in \mathcal{ID}(\boxplus)$ and has free characteristic triplet in the form:

$$\left(a, \tfrac{k(t)}{|t|}\,\mathrm{d}t, \eta\right),$$

Λ preserves selfdecomposability

Theorem [Barndorff-Nielsen+T]. For a $*$-infinitely divisible probability measure μ, we have that

$$\mu \text{ is } *\text{-sd} \iff \Lambda(\mu) \text{ is } \boxplus\text{-sd}.$$

Corollary. A probability measure ν on $\mathbb{R}$ is $\boxplus$-s.d., if and only if $\nu \in \mathcal{ID}(\boxplus)$ and has free characteristic triplet in the form:

$$\left(a, \frac{k(t)}{|t|}\,\mathrm{d}t, \eta\right),$$

where $k\colon \mathbb{R} \setminus \{0\} \to [0,\infty)$ is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$.

Unimodality

A finite measure μ on $\mathbb{R}$ is called *unimodal*, if, for some a in $\mathbb{R}$, it has the form

$$\mu(\mathrm{d}x) = \mu(\{a\})\delta_a(\mathrm{d}x) + f(x)\,\mathrm{d}x,$$

where f is increasing on $(-\infty, a)$ and decreasing on (a, ∞).

Unimodality vs. selfdecomposability – overview

Theorem [Yamasato '78]. All $*$-selfdecomposable probability measures are unimodal.

Unimodality vs. selfdecomposability – overview

Theorem [Yamasato '78]. All $*$-selfdecomposable probability measures are unimodal.

Theorem [Biane '98]. All $\boxplus$-stable probability measures are unimodal.

Unimodality vs. selfdecomposability – overview

Theorem [Yamasato '78]. All $*$-selfdecomposable probability measures are unimodal.

Theorem [Biane '98]. All ⊞-stable probability measures are unimodal.

Theorem [Haagerup+T '11]. The free gamma distributions are unimodal.

Unimodality vs. selfdecomposability – overview

Theorem [Yamasato '78]. All $*$-selfdecomposable probability measures are unimodal.

Theorem [Biane '98]. All ⊞-stable probability measures are unimodal.

Theorem [Haagerup+T '11]. The free gamma distributions are unimodal.

Theorem [Hasebe+T '13]. All freely selfdecomposable distributions are unimodal.

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Then for w in $\mathbb{C}^-$, we have that

$$\mathcal{C}_\nu(w) = w\int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{1-wt}\,\mathrm{d}t.$$

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Then for w in $\mathbb{C}^-$, we have that

$$\mathcal{C}_\nu(w) = w\int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{1 - wt}\,\mathrm{d}t.$$

Setting $w = \frac{1}{z}$ leads to

$$\mathcal{C}_\nu\left(\tfrac{1}{z}\right) = \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z - t}\,\mathrm{d}t,$$

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Then for w in $\mathbb{C}^-$, we have that

$$\mathcal{C}_\nu(w) = w \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{1 - wt}\,\mathrm{d}t.$$

Setting $w = \frac{1}{z}$ leads to

$$\frac{1}{z} G_\nu^{\langle -1 \rangle}\left(\tfrac{1}{z}\right) - 1 = \mathcal{C}_\nu\left(\tfrac{1}{z}\right) = \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z - t}\,\mathrm{d}t,$$

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Then for w in $\mathbb{C}^-$, we have that

$$\mathcal{C}_\nu(w) = w \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{1 - wt}\,\mathrm{d}t.$$

Setting $w = \frac{1}{z}$ leads to

$$\frac{1}{z} G_\nu^{\langle -1 \rangle}(\tfrac{1}{z}) - 1 = \mathcal{C}_\nu(\tfrac{1}{z}) = \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z - t}\,\mathrm{d}t,$$

and hence

$$G_\nu^{\langle -1 \rangle}(\tfrac{1}{z}) = z + z \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z - t}\,\mathrm{d}t := H_k(z).$$

Sketch of proof of unimodality

Assume $\nu \in \mathcal{L}(\boxplus)$, and $\nu \sim (0, \frac{k(t)}{|t|}\,\mathrm{d}t, \int_{-1}^{1} \mathrm{sign}(t)k(t)\,\mathrm{d}t)$.

Then for w in $\mathbb{C}^-$, we have that

$$\mathcal{C}_\nu(w) = w\int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{1-wt}\,\mathrm{d}t.$$

Setting $w = \frac{1}{z}$ leads to

$$\frac{1}{z}G_\nu^{\langle -1\rangle}(\tfrac{1}{z}) - 1 = \mathcal{C}_\nu(\tfrac{1}{z}) = \int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z-t}\,\mathrm{d}t,$$

and hence

$$G_\nu^{\langle -1\rangle}(\tfrac{1}{z}) = z + z\int_{\mathbb{R}} \frac{\mathrm{sign}(t)k(t)}{z-t}\,\mathrm{d}t := H_k(z).$$

We conclude that

$$\frac{1}{z} = G_\nu(H_k(z)) \quad \text{for alle } z \text{ in } \mathbb{C}^+ \text{ such that } H_k(z) \in \mathbb{C}^+.$$

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\tfrac{1}{\pi}\,\mathrm{Im}\left(\tfrac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Here

$$\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

for a continuous function $v_k \colon \mathbb{R} \to [0, \infty)$.

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\tfrac{1}{\pi}\,\mathrm{Im}\left(\tfrac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Here

$$\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

for a continuous function $v_k \colon \mathbb{R} \to [0, \infty)$.

It follows that that ν is absolutely continuous with density given by

$$f_\nu(H_k(x+\mathrm{i}v_k(x))) = -\frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{x + \mathrm{i}v_k(x)}\right]$$

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\tfrac{1}{\pi}\,\mathrm{Im}\left(\tfrac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Here

$$\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

for a continuous function $v_k \colon \mathbb{R} \to [0, \infty)$.

It follows that that ν is absolutely continuous with density given by

$$f_\nu(H_k(x+\mathrm{i}v_k(x))) = -\frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{x + \mathrm{i}v_k(x)}\right] = \frac{1}{\pi}\frac{v_k(x)}{x^2 + v_k(x)^2}, \quad (x \in \mathbb{R}).$$

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\tfrac{1}{\pi}\,\mathrm{Im}\left(\tfrac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Here

$$\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

for a continuous function $v_k \colon \mathbb{R} \to [0, \infty)$.

It follows that that ν is absolutely continuous with density given by

$$f_\nu(H_k(x+\mathrm{i}v_k(x))) = -\frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{x + \mathrm{i}v_k(x)}\right] = \frac{1}{\pi}\frac{v_k(x)}{x^2 + v_k(x)^2}, \quad (x \in \mathbb{R}).$$

From this expression one may argue that the equation:

$$\gamma = f_\nu(\xi)$$

Sketch of proof of unimodality (continued)

By Stieltjes Inversion the density f_ν is given by

$$f_\nu(H_k(z)) = -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{z}\right), \quad \text{whenever } H_k(z) \in \mathbb{R}.$$

Here

$$\{z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R}\} = \{x + \mathrm{i}v_k(x) \mid x \in \mathbb{R}\}.$$

for a continuous function $v_k \colon \mathbb{R} \to [0, \infty)$.

It follows that that ν is absolutely continuous with density given by

$$f_\nu(H_k(x+\mathrm{i}v_k(x))) = -\frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{x + \mathrm{i}v_k(x)}\right] = \frac{1}{\pi}\frac{v_k(x)}{x^2 + v_k(x)^2}, \quad (x \in \mathbb{R}).$$

From this expression one may argue that the equation:

$$\gamma = f_\nu(\xi)$$

has at most 2 solutions in ξ for any $\gamma > 0$.

The classical Gaussian distribution in free probability

Question [V. Perez-Abreu]: Is $N(0,1)$ infinitely divisible with respect to $\boxplus$?

The classical Gaussian distribution in free probability

Question [V. Perez-Abreu]: Is $N(0,1)$ infinitely divisible with respect to $\boxplus$?

Theorem [Belinschi, Bozejko, Lehner & Speicher, 2011]: Yes!

The classical Gaussian distribution in free probability

Question [V. Perez-Abreu]: Is $N(0,1)$ infinitely divisible with respect to $\boxplus$?

Theorem [Belinschi, Bozejko, Lehner & Speicher, 2011]: Yes!

Question [M. Bozejko]: Is $N(0,1)$ selfdecomposable with respect to $\boxplus$?

The classical Gaussian distribution in free probability

Question [V. Perez-Abreu]: Is $N(0,1)$ infinitely divisible with respect to $\boxplus$?

Theorem [Belinschi, Bozejko, Lehner & Speicher, 2011]: Yes!

Question [M. Bozejko]: Is $N(0,1)$ selfdecomposable with respect to $\boxplus$?

Theorem [Hasebe, Sakuma, T, 2016]: Yes!

Characterization of $\mathcal{ID}(\boxplus)$

Theorem [Bercovici-Voiculescu, 1993]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

Characterization of $\mathcal{ID}(\boxplus)$

Theorem [Bercovici-Voiculescu, 1993]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{ID}(\boxplus)$.

Characterization of $\mathcal{ID}(\boxplus)$

Theorem [Bercovici-Voiculescu, 1993]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{ID}(\boxplus)$.

(ii) $\mathcal{C}_\mu$ may be extended to an analytic function $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$.

Characterization of $\mathcal{ID}(\boxplus)$

Theorem [Bercovici-Voiculescu, 1993]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{ID}(\boxplus)$.

(ii) $\mathcal{C}_\mu$ may be extended to an analytic function $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$.

(iii) There exist a in $[0,\infty)$, η in $\mathbb{R}$ and a Lévy measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}_\mu(z) = \eta z + a z^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz 1_{[-1,1]}(t) \Big) \, \rho(\mathrm{d}t), \quad (z \in \mathbb{C}^-).$$

An analogous characterization of $\mathcal{L}(\boxplus)$

Theorem [Hasebe, Sakuma, T, 2016]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{L}(\boxplus)$.

An analogous characterization of $\mathcal{L}(\boxplus)$

Theorem [Hasebe, Sakuma, T, 2016]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{L}(\boxplus)$.

(ii) $\mathcal{C}_\mu$ may be extended to an analytic function $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$, such that $\mathrm{Im}(\mathcal{C}'_\mu(z)) \leq 0$ for all z in $\mathbb{C}^-$.

An analogous characterization of $\mathcal{L}(\boxplus)$

Theorem [Hasebe, Sakuma, T, 2016]: For a probability measure μ on $\mathbb{R}$, the following are equivalent:

(i) $\mu \in \mathcal{L}(\boxplus)$.

(ii) $\mathcal{C}_\mu$ may be extended to an analytic function $\mathcal{C}_\mu\colon \mathbb{C}^- \to \mathbb{C}$, such that $\mathrm{Im}(\mathcal{C}_\mu'(z)) \le 0$ for all z in $\mathbb{C}^-$.

(iii) There exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that $\int_{\mathbb{R}} \ln(2+|x|)\,\rho(\mathrm{d}x) < \infty$, and

$$\mathcal{C}_\mu'(z) = \xi + \int_{\mathbb{R}} \frac{x+z}{1-xz}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}'_\mu(z)) \leq 0$ for all z in $\mathbb{C}^-$.

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu\colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}_\mu'(z)) \leq 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}_\mu'(\frac{1}{w})\colon \mathbb{C}^+ \to \mathbb{C}^+$.

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}_\mu'(z)) \le 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}_\mu'(\frac{1}{w}) \colon \mathbb{C}^+ \to \mathbb{C}^+$.

By Nevanlinna-Pick representation, there exist a in $[0,\infty)$, ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$-\mathcal{C}_\mu'(\tfrac{1}{w}) = az - \xi + \int_{\mathbb{R}} \frac{1+xw}{x-w}\,\rho(\mathrm{d}x), \qquad (w \in \mathbb{C}^+),$$

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu \colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}_\mu'(z)) \le 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}_\mu'(\frac{1}{w}) \colon \mathbb{C}^+ \to \mathbb{C}^+$.

By Nevanlinna-Pick representation, there exist a in $[0,\infty)$, ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$-\mathcal{C}_\mu'(\tfrac{1}{w}) = az - \xi + \int_{\mathbb{R}} \frac{1+xw}{x-w}\,\rho(\mathrm{d}x), \qquad (w \in \mathbb{C}^+),$$

i.e.

$$\mathcal{C}_\mu'(z) = -\frac{a}{z} + \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu\colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}'_\mu(z)) \leq 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}'_\mu(\frac{1}{w})\colon \mathbb{C}^+ \to \mathbb{C}^+$.

By Nevanlinna-Pick representation, there exist a in $[0,\infty)$, ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$-\mathcal{C}'_\mu(\tfrac{1}{w}) = az - \xi + \int_{\mathbb{R}} \frac{1+xw}{x-w}\,\rho(\mathrm{d}x), \qquad (w \in \mathbb{C}^+),$$

i.e.

$$\mathcal{C}'_\mu(z) = -\frac{a}{z} + \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Since

$$\lim_{y\downarrow 0} \mathcal{C}_\mu(\mathrm{i}y) = 0,$$

Sketch of proof of (ii) ⇒ (iii)

Assume that $\mathcal{C}_\mu\colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}_\mu'(z)) \le 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}_\mu'(\frac{1}{w})\colon \mathbb{C}^+ \to \mathbb{C}^+$.

By Nevanlinna-Pick representation, there exist a in $[0,\infty)$, ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$-\mathcal{C}_\mu'(\tfrac{1}{w}) = az - \xi + \int_{\mathbb{R}} \frac{1+xw}{x-w}\,\rho(\mathrm{d}x), \qquad (w \in \mathbb{C}^+),$$

i.e.

$$\mathcal{C}_\mu'(z) = -\frac{a}{z} + \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Since

$$\lim_{y\downarrow 0} \mathcal{C}_\mu(\mathrm{i}y) = 0, \quad \text{and} \quad \mathcal{C}_\mu(\mathrm{i}y) = \mathcal{C}_\mu(-\mathrm{i}) - \mathrm{i}\int_0^y \mathcal{C}_\mu'(-\mathrm{i}t)\,\mathrm{d}t,$$

Sketch of proof of (ii) $\Rightarrow$ (iii)

Assume that $\mathcal{C}_\mu\colon \mathbb{C}^- \to \mathbb{C}$ is analytic such that $\mathrm{Im}(\mathcal{C}'_\mu(z)) \le 0$ for all z in $\mathbb{C}^-$.

Consider the analytic function $w \mapsto -\mathcal{C}'_\mu(\frac{1}{w})\colon \mathbb{C}^+ \to \mathbb{C}^+$.

By Nevanlinna-Pick representation, there exist a in $[0,\infty)$, ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$-\mathcal{C}'_\mu(\tfrac{1}{w}) = az - \xi + \int_{\mathbb{R}} \frac{1+xw}{x-w}\,\rho(\mathrm{d}x), \qquad (w \in \mathbb{C}^+),$$

i.e.

$$\mathcal{C}'_\mu(z) = -\frac{a}{z} + \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Since

$$\lim_{y\downarrow 0} \mathcal{C}_\mu(\mathrm{i}y) = 0, \quad \text{and} \quad \mathcal{C}_\mu(\mathrm{i}y) = \mathcal{C}_\mu(-\mathrm{i}) - \mathrm{i}\int_0^y \mathcal{C}'_\mu(-\mathrm{i}t)\,\mathrm{d}t,$$

one may argue that $a = 0$, and $\int_{\mathbb{R}} \ln(2+|x|)\,\rho(\mathrm{d}x) < \infty$.

Sketch of proof of (iii) $\Rightarrow$ (i)

Assume that there exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}'_\mu(z) = \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Sketch of proof of (iii) ⇒ (i)

Assume that there exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}'_\mu(z) = \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Then for any z in $\mathbb{C}^-$ we find that

$$\mathcal{C}_\mu(z) = \mathcal{C}_\mu(-\mathrm{i}) + \int_{-i}^{z} \mathcal{C}'_\mu(\omega)\,\mathrm{d}\omega$$

Sketch of proof of (iii) ⇒ (i)

Assume that there exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}_\mu'(z) = \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Then for any z in $\mathbb{C}^-$ we find that

$$\begin{aligned}\mathcal{C}_\mu(z) &= \mathcal{C}_\mu(-\mathrm{i}) + \int_{-i}^{z} \mathcal{C}_\mu'(\omega)\,\mathrm{d}\omega \\ &= \mathcal{C}_\mu(-\mathrm{i}) + \xi(z+\mathrm{i}) + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{-\mathrm{i}}^{z} \frac{x+\omega}{1-x\omega}\,\mathrm{d}\omega \Big)\,\rho(\mathrm{d}x)\end{aligned}$$

Sketch of proof of (iii) $\Rightarrow$ (i)

Assume that there exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}'_\mu(z) = \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Then for any z in $\mathbb{C}^-$ we find that

$$\mathcal{C}_\mu(z) = \mathcal{C}_\mu(-\mathrm{i}) + \int_{-i}^{z} \mathcal{C}'_\mu(\omega)\,\mathrm{d}\omega$$

$$= \mathcal{C}_\mu(-\mathrm{i}) + \xi(z+\mathrm{i}) + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{-\mathrm{i}}^{z} \frac{x+\omega}{1-x\omega}\,\mathrm{d}\omega \Big)\,\rho(\mathrm{d}x)$$

where $c = \frac{1}{2}\rho(\{0\})$.

Sketch of proof of (iii) ⇒ (i)

Assume that there exist ξ in $\mathbb{R}$ and a measure ρ on $\mathbb{R}$, such that

$$\mathcal{C}'_\mu(z) = \xi + \int_{\mathbb{R}} \frac{z+x}{1-zx}\,\rho(\mathrm{d}x), \qquad (z \in \mathbb{C}^-).$$

Then for any z in $\mathbb{C}^-$ we find that

$$\begin{aligned}
\mathcal{C}_\mu(z) &= \mathcal{C}_\mu(-\mathrm{i}) + \int_{-i}^{z} \mathcal{C}'_\mu(\omega)\,\mathrm{d}\omega \\
&= \mathcal{C}_\mu(-\mathrm{i}) + \xi(z+\mathrm{i}) + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(\int_{-\mathrm{i}}^{z} \frac{x+\omega}{1-x\omega}\,\mathrm{d}\omega \Big)\,\rho(\mathrm{d}x) \\
&= A + \xi z + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(-\log(1-xz) - \frac{xz}{1+x^2} \Big) \frac{1+x^2}{x^2}\,\rho(\mathrm{d}x),
\end{aligned}$$

where $c = \frac{1}{2}\rho(\{0\})$.

Sketch of proof of (iii) ⇒ (i) (continued)

Now put

$$k(x) = \begin{cases} \int_x^\infty \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x > 0, \\ \int_{-\infty}^x \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x < 0. \end{cases}$$

Sketch of proof of (iii) ⇒ (i) (continued)

Now put

$$k(x) = \begin{cases} \int_x^\infty \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x > 0, \\ \int_{-\infty}^x \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x < 0. \end{cases}$$

The assumptions on ρ ensure that $\frac{k(x)}{|x|}\,\mathrm{d}x$ is a Lévy measure.

Sketch of proof of (iii) ⇒ (i) (continued)

Now put

$$k(x) = \begin{cases} \int_x^\infty \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x > 0, \\ \int_{-\infty}^x \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x < 0. \end{cases}$$

The assumptions on ρ ensure that $\frac{k(x)}{|x|}\,\mathrm{d}x$ is a Lévy measure.

Using integration by parts, we further obtain that

$$\mathcal{C}_\mu(z) = A + \xi z + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(-\log(1 - xz) - \frac{xz}{1+x^2} \Big) \frac{1+x^2}{x^2}\,\rho(\mathrm{d}x)$$

Sketch of proof of (iii) ⇒ (i) (continued)

Now put

$$k(x) = \begin{cases} \int_x^\infty \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x > 0, \\ \int_{-\infty}^x \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x < 0. \end{cases}$$

The assumptions on ρ ensure that $\frac{k(x)}{|x|}\,\mathrm{d}x$ is a Lévy measure.

Using integration by parts, we further obtain that

$$\mathcal{C}_\mu(z) = A + \xi z + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(-\log(1-xz) - \frac{xz}{1+x^2}\Big)\frac{1+x^2}{x^2}\,\rho(\mathrm{d}x)$$

$$= A + \tilde{\xi} z + cz^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-xz} - 1 - xz1_{[-1,1]}(x)\Big)\frac{k(x)}{|x|}\,\mathrm{d}x.$$

Sketch of proof of (iii) ⇒ (i) (continued)

Now put

$$k(x) = \begin{cases} \int_x^\infty \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x > 0,\\ \int_{-\infty}^x \frac{1+t^2}{t^2}\,\rho(\mathrm{d}t), & \text{if } x < 0. \end{cases}$$

The assumptions on ρ ensure that $\frac{k(x)}{|x|}\,\mathrm{d}x$ is a Lévy measure.

Using integration by parts, we further obtain that

$$\mathcal{C}_\mu(z) = A + \xi z + cz^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(-\log(1-xz) - \frac{xz}{1+x^2}\Big)\frac{1+x^2}{x^2}\,\rho(\mathrm{d}x)$$

$$= A + \tilde{\xi} z + cz^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-xz} - 1 - xz1_{[-1,1]}(x)\Big)\frac{k(x)}{|x|}\,\mathrm{d}x.$$

Since $\mathcal{C}_\mu(-\mathrm{i}y) \to 0$, as $y \downarrow 0$, we must have that $A = 0$.

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu \colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneq \Omega$.

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu \colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneqq \Omega$.

(b) $$\frac{F'_\mu(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$$

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu \colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneqq \Omega$.

(b) $\dfrac{F'_\mu(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$

Now for any z in $\mathbb{C}^-$, we put $\omega = F_\mu^{-1}(\frac{1}{z}) \in \Omega$.

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu \colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneq \Omega$.

(b) $$\frac{F_\mu'(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$$

Now for any z in $\mathbb{C}^-$, we put $\omega = F_\mu^{-1}(\frac{1}{z}) \in \Omega$.

Then since $\mathcal{C}_\mu(z) = zF_\mu^{-1}(\frac{1}{z}) - 1$, we find that

$$\mathcal{C}_\mu'(z) = F_\mu^{-1}(\tfrac{1}{z}) - \frac{1}{z}\frac{1}{F_\mu'(F_\mu^{-1}(\frac{1}{z}))}$$

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu}\colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu\colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneq \Omega$.

(b) $$\frac{F'_\mu(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$$

Now for any z in $\mathbb{C}^-$, we put $\omega = F_\mu^{-1}(\frac{1}{z}) \in \Omega$.

Then since $\mathcal{C}_\mu(z) = zF_\mu^{-1}(\frac{1}{z}) - 1$, we find that

$$\mathcal{C}'_\mu(z) = F_\mu^{-1}(\tfrac{1}{z}) - \frac{1}{z}\frac{1}{F'_\mu(F_\mu^{-1}(\frac{1}{z}))} = \omega - \frac{F_\mu(\omega)}{F'_\mu(\omega)}$$

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu} \colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu \colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneqq \Omega$.

(b) $$\frac{F'_\mu(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$$

Now for any z in $\mathbb{C}^-$, we put $\omega = F_\mu^{-1}(\frac{1}{z}) \in \Omega$.

Then since $\mathcal{C}_\mu(z) = zF_\mu^{-1}(\frac{1}{z}) - 1$, we find that

$$\mathcal{C}'_\mu(z) = F_\mu^{-1}(\tfrac{1}{z}) - \frac{1}{z}\frac{1}{F'_\mu(F_\mu^{-1}(\frac{1}{z}))} = \omega - \frac{F_\mu(\omega)}{F'_\mu(\omega)} = \omega - \frac{1}{\omega - F_\mu(\omega)}.$$

Sketch of proof of: $\mu := N(0,1) \in \mathcal{L}(\boxplus)$

From the work of Belinschi, Bozejko, Lehner and Speicher we have

(a) $F_\mu = \frac{1}{G_\mu}\colon \mathbb{C}^+ \to \mathbb{C}^+$, may be extended to an analytic bijection $F_\mu\colon \Omega \to \mathbb{C}^+$, where Ω is an open connected set, and $\mathbb{C}^+ \subsetneqq \Omega$.

(b) $\dfrac{F'_\mu(\omega)}{F_\mu(\omega)} = \omega - F_\mu(\omega), \qquad (\omega \in \Omega).$

Now for any z in $\mathbb{C}^-$, we put $\omega = F_\mu^{-1}(\frac{1}{z}) \in \Omega$.

Then since $\mathcal{C}_\mu(z) = zF_\mu^{-1}(\frac{1}{z}) - 1$, we find that

$$\mathcal{C}'_\mu(z) = F_\mu^{-1}(\tfrac{1}{z}) - \frac{1}{z}\frac{1}{F'_\mu(F_\mu^{-1}(\frac{1}{z}))} = \omega - \frac{F_\mu(\omega)}{F'_\mu(\omega)} = \omega - \frac{1}{\omega - F_\mu(\omega)}.$$

So it remains to argue that

$$\operatorname{Im}\Big(\omega - \frac{1}{\omega - F_\mu(\omega)}\Big) \le 0, \qquad (\omega \in \Omega).$$