Modelling multivariate serially correlated count data in continuous time

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Conference on Ambit Fields and Related Topics
Aarhus University, 15-18 August 2016

- Modelling **multivariate time series of counts**.
- Count data: Non-negative and integer-valued, and often over-dispersed (i.e. variance $>$ mean).
- Various applications: Medical science, epidemiology, meteorology, network modelling, actuarial science, econometrics and finance.

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Aim of the project

Develop continuous-time models for time series of counts that

- allows for a flexible autocorrelation structure;
- can deal with a variety of marginal distributions;
- allows for flexibility when modelling cross-correlations;
- can cope with asynchronous, non-equidistant observations;
- is analytically tractable.

- Overall, two predominant modelling approaches:
  - Discrete autoregressive moving-average (DARMA) models introduced by Jacobs & Lewis (1978a,b).

    The advantage of such stationary processes is that their marginal distribution can be of any kind. However, this comes at the cost that the dependence structure is generated by potentially long runs of constant values, which results in sample paths which are rather unrealistic in many applications (see McKenzie (2003)).
  - Models obtained from thinning operations going back to the influential work of Steutel & van Harn (1979), e.g. INARMA. See also Zhu & Joe (2003) for related more recent work.

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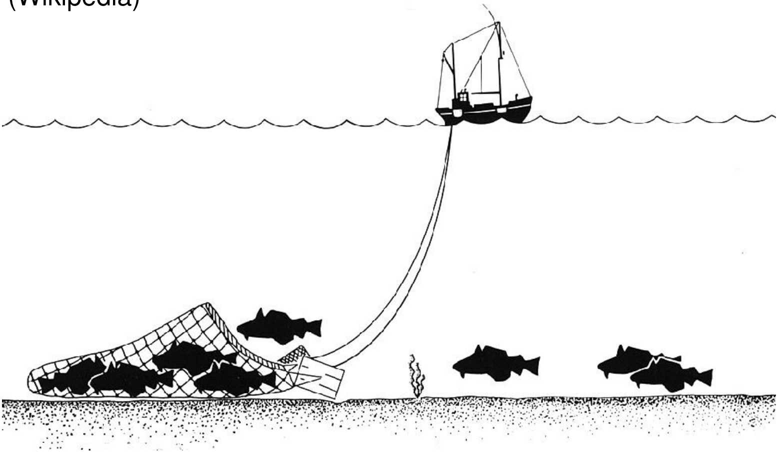
  - Models obtained from thinning operations going back to the influential work of Steutel & van Harn (1979), e.g. INARMA. See also Zhu & Joe (2003) for related more recent work.
- Key idea of this paper: Use trawling for modelling counts! – Nested within the framework of ambit fields (Barndorff-Nielsen & Schmiegel (2007)) and extends results by Barndorff-Nielsen, Pollard & Shephard (2012) and Barndorff-Nielsen, Lunde, Shephard & Veraart (2014).

Introduction

What is trawling...? A first "definition"

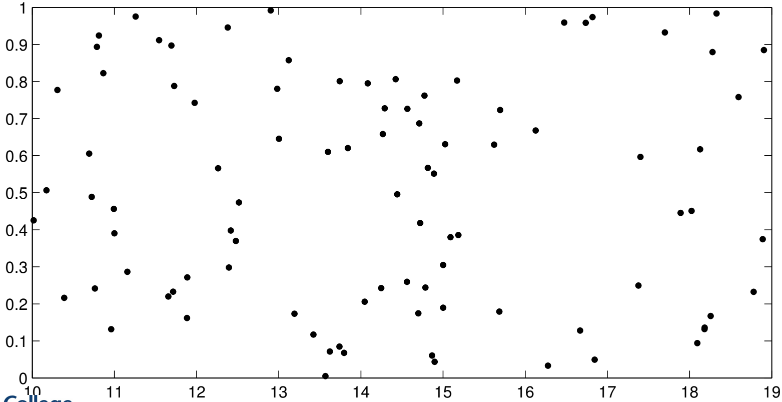
"Trawling is a method of fishing that involves pulling a fishing net through the water behind one or more boats. The net that is used for trawling is called a trawl." (Wikipedia)



- Let N be a homogeneous Poisson random measure on $\mathbb{R}^n \times \mathbb{R}^2$ with compensator
$$\mathbb{E}(N(d\mathbf{y}, dx, dt)) = \nu(d\mathbf{y})dxdt,$$
where ν is a Lévy measure satisfying $\int_{-\infty}^{\infty} \min(1, ||\mathbf{y}||)\nu(d\mathbf{y}) < \infty$.
- Assume that N is positive integer-valued, i.e. ν is concentrated on $\mathbb{N}$.
- Then we define an $\mathbb{N}^n$-valued, homogeneous Lévy basis on $\mathbb{R}^2$ in terms of the Poisson random measure as
$$\mathbf{L}(dx, dt) = (L^{(1)}(dx, ds), \ldots, L^{(n)}(dx, ds))^\top = \int_{-\infty}^{\infty} \mathbf{y} N(d\mathbf{y}, dx, dt). \quad (1)$$
- The Lévy basis $\mathbf{L}$ is infinitely divisible with cumulant function
$$C(\theta \ddagger \mathbf{L}(dx, dt)) := \log(\mathbb{E}(\exp(i\theta^\top L(dx, dt)))) = \int_{\mathbb{R}} \left(e^{i\theta^\top \mathbf{y}} - 1\right) \nu(d\mathbf{y})dxdt$$
$$= C(\theta \ddagger \mathbf{L}')dxdt, \text{ where } \mathbf{L}' \text{ is the Lévy seed.}$$

Theoretical framework
Example: A Poisson Basis

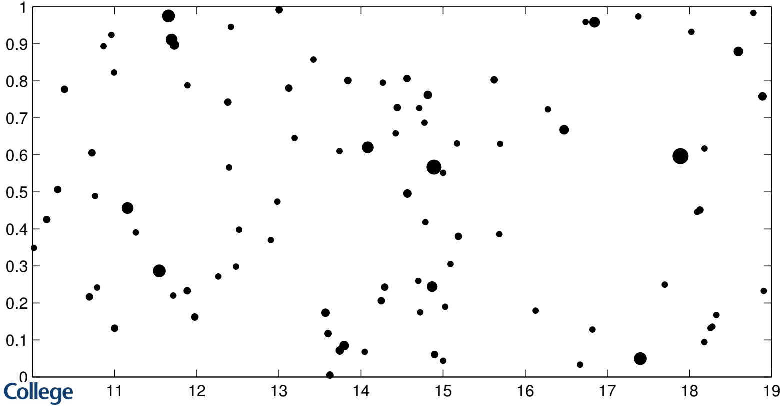
- Suppose $L^{(i)}(dx, dt)$ is a Poisson basis (Poisson random measure) on $[0, 1] \times \mathbb{R}_+$ with mean $\upsilon dxdt$.
  - $L^{(i)}$ generates randomly scattered points in time with uniformly distributed height over a unit height strip.



Theoretical framework

Example: A Negative Binomial Basis

- Say we want the negative binomial Lévy seed ($L'_t \sim NB(v^{(i)}t, \theta^{(i)})$)
  - Then use $L'^{(i)}_t = \sum_{j=1}^{N^{(i)}_t} C^{(i)}_j$ where $N^{(i)} \sim Poi(v \left|\log\left(1-\theta^{(i)}\right)\right|)$ and $C^{(i)}_j$ (integer dot size) follow the logarithmic distribution with parameter $\theta^{(i)}$.



- From Feller (1968), we know that any non-degenerate distribution on $\mathbb{N}^n$ is infinitely divisible if and only if it can be expressed as a discrete compound Poisson distribution.
- A random vector with infinitely divisible distribution on $\mathbb{N}^n$ always has non-negatively correlated components.
- We model the Lévy seed by an n-dimensional compound Poisson process given by

$$\mathbf{L}'_t = \sum_{j=1}^{N_t} \mathbf{Z}_j,$$

where $N = (N_t)_{t \geq 0}$ is an homogeneous Poisson process of rate $v > 0$ and the $(\mathbf{Z}_j)_{j \in \mathbb{N}}$ form a sequence of i.i.d. random variables independent of N and which have no atom in $\mathbf{0}$, i.e. not all components are simultaneously equal to zero, more precisely, $\mathbb{P}(\mathbf{Z}_j = \mathbf{0}) = 0$ for all j.

Definition 1

A trawl for the *i*th component is a Borel set $A^{(i)} \subset \mathbb{R} \times (-\infty, 0]$ such that $\mathrm{Leb}(A^{(i)}) < \infty$. Then, we set

$$A_t^{(i)} = A^{(i)} + (0, t), \qquad i \in \{1, \ldots, n\}.$$

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$$A_t^{(i)} = A^{(i)} + (0, t), \qquad i \in \{1, \ldots, n\}.$$

- Typically, we choose $A^{(i)}$ to be of the form

$$A^{(i)} = \{(x, s) : s \leq 0,\ 0 \leq x \leq d^{(i)}(s)\}, \tag{2}$$

  where $d^{(i)} : (-\infty, 0] \mapsto \mathbb{R}$ is a cont. and $\mathrm{Leb}(A^{(i)}) < \infty$.
- Then $\quad A_t^{(i)} = A^{(i)} + (0, t) = \{(x, s) : s \leq t,\ 0 \leq x \leq d^{(i)}(s - t)\}$.
- If $d^{(i)}$ is also monotonically non-decreasing, then $A^{(i)}$ is a *monotonic trawl*.

Definition 2

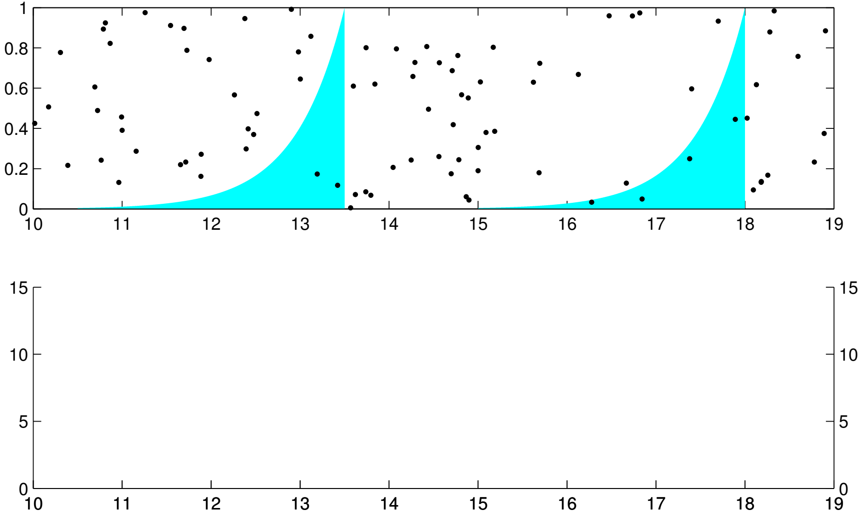
We define an *n-dimensional stationary integer-valued trawl (IVT) process* $(\mathbf{Y}_t)_{t\geq 0}$ by $\mathbf{Y}_t = (L^{(1)}(A_t^{(1)}), \ldots, L^{(n)}(A_t^{(n)}))'$,

- where $$L^{(i)}(A_t^{(i)}) = \int_{\mathbb{R}\times\mathbb{R}} \mathbf{I}_{A^{(i)}}(x, s-t) L^{(i)}(dx, ds), \quad i \in \{1, \ldots, n\}.$$
- $\mathbf{L}$ is the n-dimensional integer-valued, homogeneous Lévy basis on $\mathbb{R}^2$ (see (1)).
- $A_t^{(i)} = A^{(i)} + (0, t)$ with $A^{(i)} \subset \mathbb{R} \times (-\infty, 0]$ and $\mathrm{Leb}(A^{(i)}) < \infty$ is the trawl.

- Wolpert & Taqqu (2005) study a subclass of general (univariate) trawl processes (not necessarily restricted to IV) under the name “up-stairs” representation, “random measure of a moving geometric figure in a higher-dimensional space”
- Wolpert & Brown (2011) study so-called “random measure processes” which also fall into the (univariate) trawling framework.

Theoretical framework

Definition of a Trawl Process



Theoretical framework

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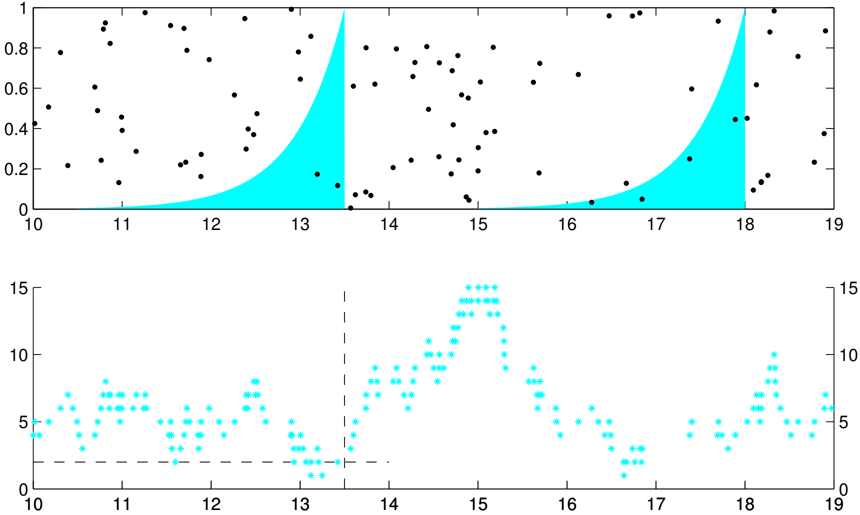
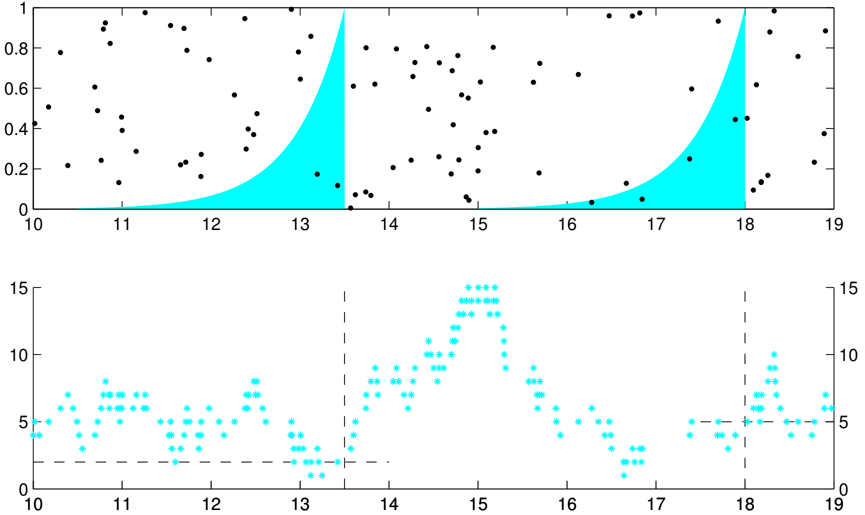


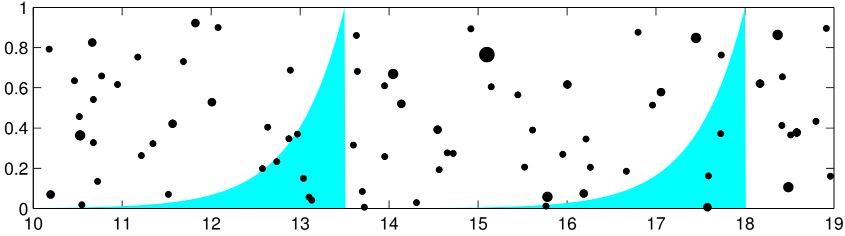
Theoretical framework

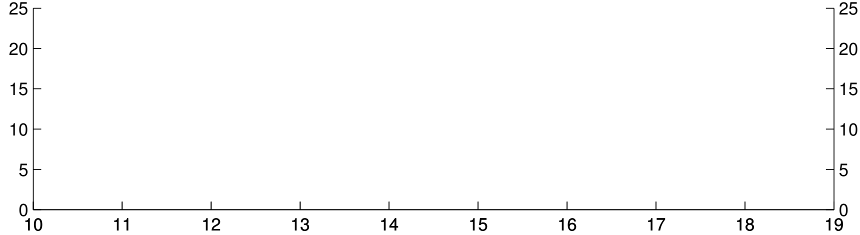
Definition of a Trawl Process



Examples

Negative Binomial exponential-trawl process





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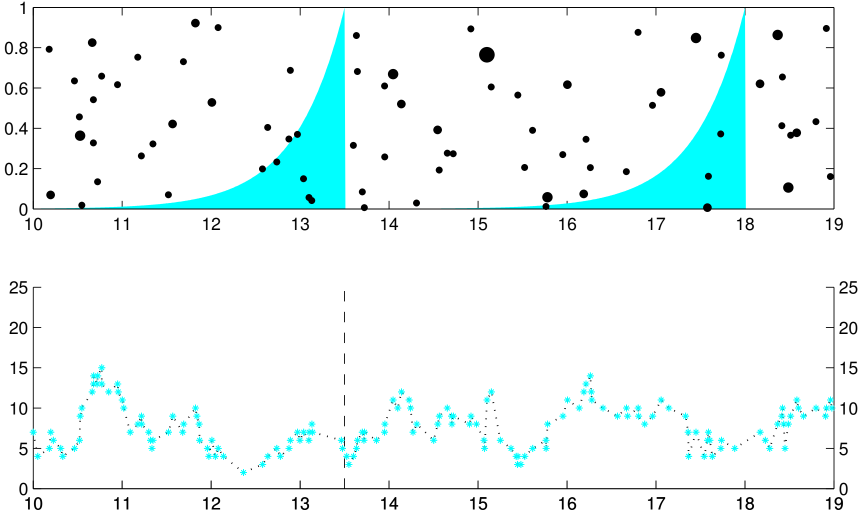

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Examples

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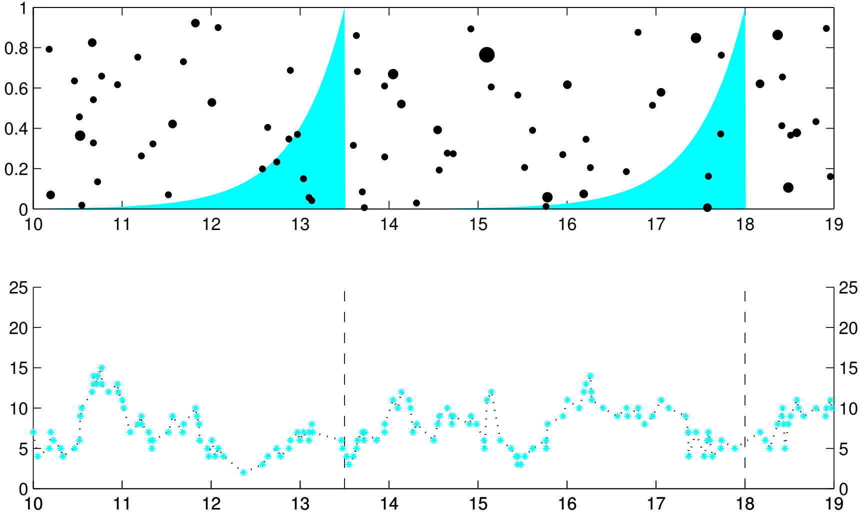

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Some key properties of IVT processes

- The IVT process is stationary and infinitely divisible.
- The IVT process is mixing $\Rightarrow$ weakly mixing $\Rightarrow$ ergodic.
- For any $\boldsymbol{\theta} \in \mathbb{R}^n$, the characteristic function of $\mathbf{Y}_t$ is given by $\mathbb{E}(\exp(i\boldsymbol{\theta}^\top \mathbf{Y}_t)) = \exp(C_{\mathbf{Y}_t}(\boldsymbol{\theta}))$, where

$$C_{\mathbf{Y}_t}(\boldsymbol{\theta}) = \sum_{k=1}^{n} \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n: \\ i_\nu \neq i_\mu, \text{ for } \nu \neq \mu}} \mathrm{Leb}\left(\bigcap_{l=1}^{k} A^{(i_l)} \setminus \bigcup_{\substack{1 \leq j \leq n, \\ j \notin \{i_1, \ldots, i_k\}}} A^{(j)} \right) C_{(L^{(i_1)}, \ldots, L^{(i_k)})}((\theta_{i_1}, \ldots, \theta_{i_k})^\top).$$

- In the special case when $A^{(1)} = \cdots = A^{(n)} = A$, the characteristic function simplifies to $\mathbb{E}(\exp(i\boldsymbol{\theta}^\top \mathbf{Y}_t)) = \exp\left(\mathrm{Leb}(A) C_{\mathbf{L}'}(\boldsymbol{\theta})\right)$.

Some key properties of IVT processes

Cross-correlation structure

- The covariance between two (possibly shifted) components $1 \leq i \leq j \leq n$ for $t, h \geq 0$ is given by

$$\rho_{ij}(h) := Cov\left(L^{(i)}(A_t^{(i)}), L^{(j)}(A_{t+h}^{(j)})\right)$$
$$= \underbrace{\mathrm{Leb}\left(A^{(i)} \cap A_h^{(j)}\right)}_{=:R_{ij}(h)} \underbrace{\left(\int_{\mathbb{R}} \int_{\mathbb{R}} y_i y_j \nu^{(i,j)}(dy_i, dy_j)\right)}_{=:\kappa_{i,j}}.$$

- Suppose the trawls $A^{(i)}$, $i \in \{1, \ldots, n\}$ are of type (2). Then for $h \geq 0$ the intersection of two trawls is given by

$$A^{(i)} \cap A_h^{(j)} = \{(x,s) : s \leq 0, 0 \leq x \leq \min\{d^{(i)}(s), d^{(j)}(s-h)\}\}.$$

I.e.

$$R_{ij}(h) = \int_{-\infty}^{0} \min\{d^{(i)}(s), d^{(j)}(s-h)\} ds.$$

Some key properties of IVT processes

Autocorrelation structure

- For each component, the autocorrelation function is given by

$$r^{(i)}(h) = Cor(L^{(i)}(A_t^{(i)}), L^{(i)}(A_{t+h}^{(i)})) = \frac{\mathrm{Leb}(A^{(i)} \cap A_h^{(i)})}{\mathrm{Leb}(A^{(i)})}.$$

- For a monotonic trawl, we get

$$\mathrm{Leb}(A^{(i)} \cap A_h^{(i)}) = \int_h^\infty d^{(i)}(-x)dx,$$

which implies that

$$r^{(i)}(h) = \frac{\int_h^\infty d^{(i)}(-x)dx}{\int_0^\infty d^{(i)}(-x)dx}, \qquad \text{and} \qquad r^{(i)\prime}(h) = \frac{-d^{(i)}(-h)}{\int_0^\infty d^{(i)}(-x)dx}.$$

Key properties of IVT processes

Autocorrelation Structure: Exponential Trawl and Superpositions

- A very flexible way of parametrising $d^{(i)}$ is to work with a superposition of exponential trawls.
- Here we randomise the memory parameter λ: So

$$d^{(i)}(z) = \int_0^\infty e^{\lambda z} f_{\pi^{(i)}}(\lambda) d\lambda, \qquad \text{for } z \leq 0,$$

  for an absolutely continuous probability measure $\pi^{(i)}$ on $(0, \infty)$ with density $f_{\pi^{(i)}}$.
- Then the autocorrelation function is given by

$$r^{(i)}(h) = Cor(Y_t^{(i)}, Y_{t+h}^{(i)}) = \frac{\int_0^\infty \frac{1}{\lambda} e^{-\lambda h} \pi^{(i)}(d\lambda)}{\int_0^\infty \frac{1}{\lambda} \pi^{(i)}(d\lambda)},$$

  when $\int_0^\infty \frac{1}{\lambda} \pi^{(i)}(d\lambda) < \infty$.

Key properties of IVT processes
Examples with short and long memory

We suppress the superscript (i) in the following:

- sup-IG trawl (for π having inverse Gaussian distribution):

$$d(z) = \left(1 - \frac{2z}{\gamma^2}\right)^{-1/2} \exp\left(\delta\gamma\left(1 - \sqrt{1 - \frac{2z}{\gamma^2}}\right)\right), \quad \delta, \gamma > 0,$$

$$r(h) = \exp\left(\delta\gamma\left(1 - \sqrt{1 + \frac{2h}{\gamma^2}}\right)\right), \quad h \geq 0.$$

- sup-Gamma trawl (for π having Gamma distribution):

$$d(z) = \left(1 - \frac{z}{\alpha}\right)^{-H}, \quad \alpha > 0, H > 1,$$

$$r(h) = \left(1 + \frac{h}{\alpha}\right)^{1-H}.$$

  - if $H \in (1, 2]$ then this is a stationary long-memory model, while
  - if $H > 2$ it is a stationary short-memory process.

Multivariate law of the Lévy seed

Poisson mixtures

- The law of $\mathbf{L}'$ is of discrete compound Poisson type by construction.
- Use Poisson mixtures based on random additive effect models, see Barndorff-Nielsen et al. (1992).
- Consider random variables $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$, such that, conditionally on $\{Z_1, \ldots, Z_n\}$, the $X_1, \ldots, X_n$ are independent and Poisson distributed with means given by the $\{Z_1, \ldots, Z_n\}$.
- Model the joint distribution of the $\{Z_1, \ldots, Z_n\}$ by a so-called additive effect model as follows:

$$Z_i = \alpha_i U + V_i, \quad i = 1, \ldots, n,$$

  where the random variables $U, V_1, \ldots, V_n$ are independent and the $\alpha_1, \ldots, \alpha_n$ are nonnegative parameters.
- We have explicit formulas for the joint law of $(X_1, \ldots, X_n)$ and

$$\mathbb{E}(X_i) = \alpha_i \mathbb{E}(U) + \mathbb{E}(V_i), \quad i = 1, \ldots, n,$$
$$\mathrm{Cov}(X_i, X_j) = \alpha_i \alpha_j \mathrm{Var}(U), \text{ if } i \neq j.$$

Proposition

Representation as compound Poisson distribution

The Poisson mixture model of random-additive-effect type can be represented as a compound Poisson distribution with rate

$$v = -\left(\overline{K}_U(\alpha) + \sum_{i=1}^{n} \overline{K}_{V_i}(1)\right),$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $\overline{K}$ denotes the kumulant function, i.e. the logarithm of the Laplace transform, and the jump size distribution has Laplace transform

$$\mathcal{L}(\boldsymbol{\theta}; \mathbf{C}) = \frac{1}{v}\left\{\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} \alpha_i e^{-\theta_i}\right)^k q_k^{(U)} + \sum_{i=1}^{n}\sum_{k=1}^{\infty} e^{-\theta_i k} q_k^{(V_i)}\right\},$$

where

$$q_k^{(U)} = \frac{1}{k!}\int_{\mathbb{R}} e^{-\alpha x} x^k \nu_U(dx), \quad q_k^{(V_i)} = \int_{\mathbb{R}} \frac{x^k}{k!} e^{-x} \nu_{V_i}(dx), \qquad \text{for } i \in \{1, \ldots, n\},$$

where the Lévy measure of U and V_i are denoted by ν_U and ν_{V_i}, respectively.

Multivariate negative binomial law

- If U and V_is follow suitable Gamma distributions, then negative binomial marginal law can be achieved allowing for 1) independence, 2) complete dependence, or 3) dependence with additional independent factors between the components.

- Focus on Case 2 (which arises in the empirical study): Choose $U \sim \Gamma(\kappa, 1)$ and $V_i \equiv 0$, for $i = 1, \ldots, n$. Then $X_i \sim NB(\kappa, \alpha_i/(1+\alpha_i))$.

- The distribution in Case 2) can be represented as a compound Poisson distribution,
  - with rate $\kappa \log(1 + \sum_{i=1}^n \alpha_i)$ and
  - jump size distribution given by the multivariate logarithmic distribution with parameters $(p_1, \ldots, p_n)$ for $p_i = \alpha_i/(1 + \sum_{i=1}^n \alpha_i)$. I.e.

$$P(\mathbf{C} = \mathbf{c}) = \frac{\Gamma(c_1 + \cdots + c_n)}{c_1! \cdots c_n!} \frac{p_1^{c_1} \cdots p_n^{c_n}}{(-\log(1 - \sum_{i=1}^n p_i))}, \quad \text{for } \mathbf{c} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}.$$

Key properties of IVT processes

Overview

Flexible marginal distributions:

- Poisson trawl process;
- Negative binomial trawl process;
- other compound Poisson distributions.

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Flexible marginal distributions:

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- other compound Poisson distributions.

Various choices of the trawl function:

- Superpositions of exponential trawls: $d^{(i)}(z) = \int_0^\infty e^{\lambda z}\pi^{(i)}(d\lambda)$, for $z \leq 0$, for a probability measure $\pi^{(i)}$ on $(0, \infty)$.
- Possibility of allowing for long memory.
- A possible seasonal model: $d^{(i)}(t) = d_p^{(i)}(t) d_s^{(i)}(t)$, where $d_p^{(i)}(t)$ monotonically increases with t while $d_s^{(i)}(t)$ is a purely periodic seasonal effect.

Simulation and inference

Simulation

- A univariate trawl process can be written as

$$Y_t = L(A_t) = X_{0,t} + X_t,$$

where $X_{0,t} = L(\{(x,s) : s \leq 0, 0 \leq x \leq d(s-t)\})$ and $X_t = L(\{(x,s) : 0 < s \leq t, 0 \leq x \leq d(s-t)\})$.

- Since $X_{0,t} \to 0$ in probability as $t \to \infty$, we focus on X_t, which can be represented as

$$X_t = \sum_{j=1}^{N_t} C_j \mathbb{I}_{\{U_j \leq d(t_j - t)\}}, \quad (3)$$

for i.i.d. standard uniform (U_j)s (independent of N, C).

- In the multivariate context, we allow for both common and disjoint jumps and obtain the marginal components as their sums.
- Use (3) for efficient simulation of trawl processes.

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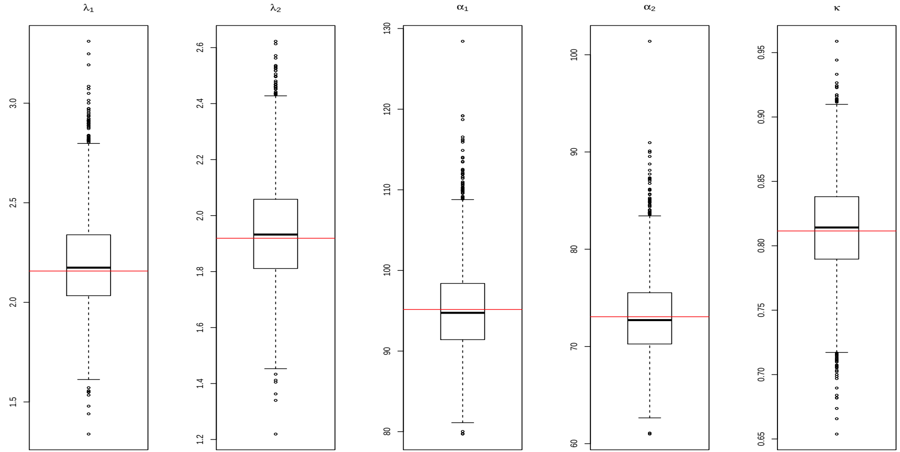
- Use a (generalised) method of moments in a two-stage equation-by-equation approach to estimate the marginal parameters first, followed by the dependence parameters.
- Step 1a) Use the acf $r^{(i)}(h) = \frac{\mathrm{Leb}(A^{(i)} \cap A_h^{(i)})}{\mathrm{Leb}(A^{(i)})}$ to infer the trawl parameters.
- Step 1b) Use the cumulant function $C(\theta \ddagger Y_t^{(i)}) = \mathrm{Leb}(A^{(i)}) C(\theta \ddagger L'^{(i)})$ to infer the marginal parameters of the Lévy basis.
- Step 2a) Compute $\mathrm{Leb}(A^{(i)} \cap A^{(j)})$ for $i \neq j$.
- Step 2b) Use the cross-covariance function

$$Cov\left(L^{(i)}(A_t^{(i)}), L^{(j)}(A_t^{(j)})\right) = \mathrm{Leb}\left(A^{(i)} \cap A^{(j)}\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} y_i y_j \nu^{(i,j)}(dy_i, dy_j)\right)$$

to infer the dependence parameters.

Simulation results

Bivariate negative binomial marginal law, with exponential trawl

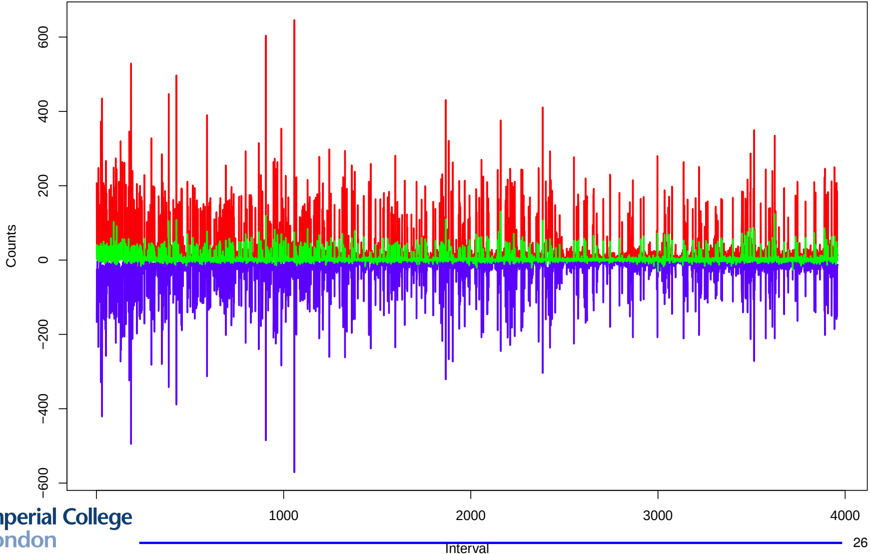


5000 Monte Carlo runs, where each sample contains 3960 observations.

- Study high frequency *limit order book data* from LOBSTER.
- We picked the Bank of America (BAC) data for 21st April 2016: Start at 10:00, end at 15:30, i.e. we removed the first and last 30 minutes.
- We compute the number of new submissions and full cancellations of limit orders in each interval of length 5s (3960 observations in total)
- We fit a bivariate trawl model to the submitted and cancelled orders.
- Summary statistics:

	Min	1st Quartile	Median	Mean	3rd Quartile	Max
No. of sub.	0	7	13	34.06	28	646
No. of can.	0	5.75	12	29.13	27.25	571

Number of submitted and cancelled limit orders



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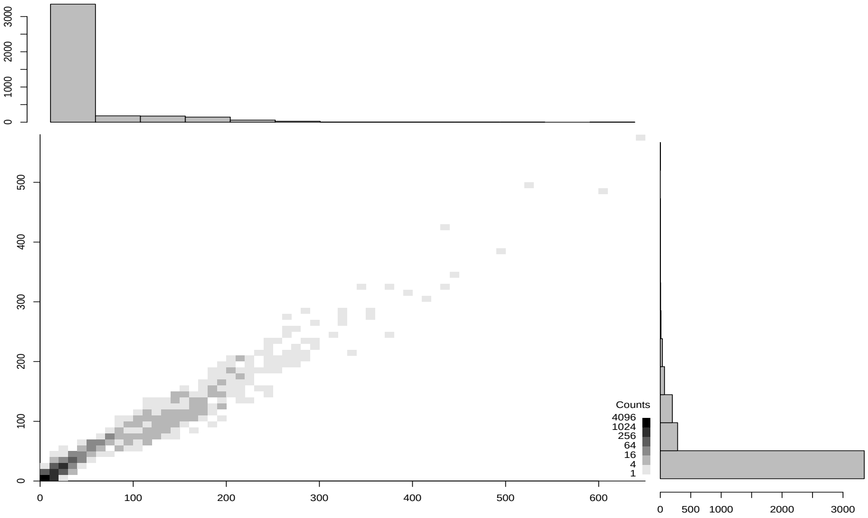

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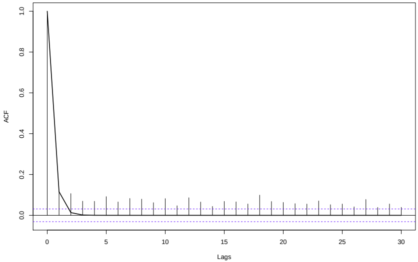
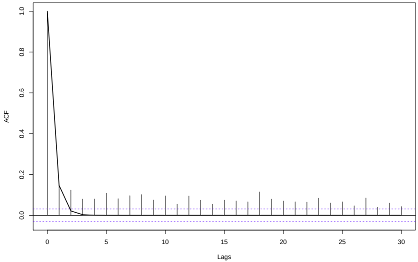
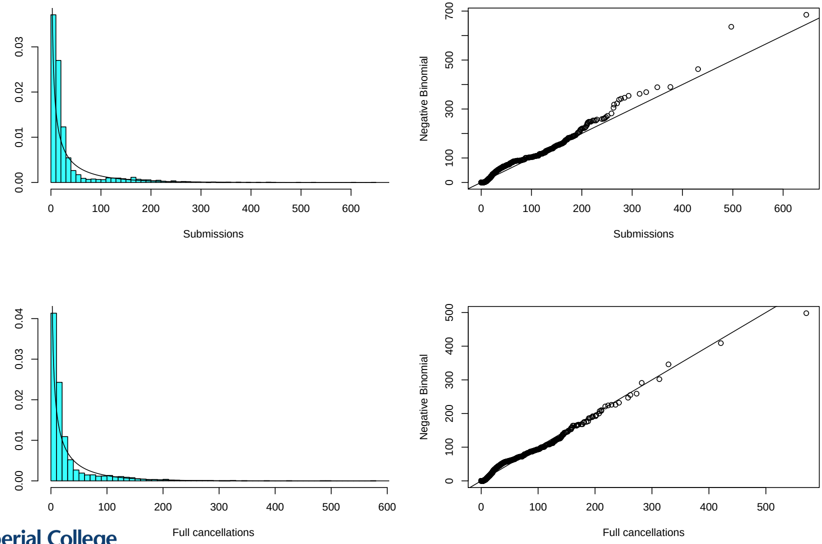
Fitted trawl functions

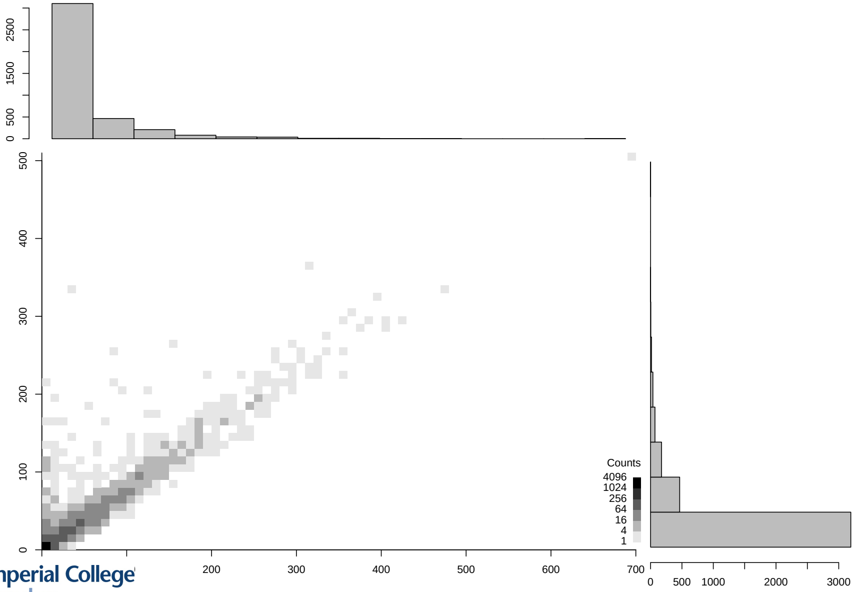




Negative binomial marginal fit



Negative binomial bivariate fit



Main contributions

- New continuous-time framework for modelling multivariate stationary, serially correlated count data.
- Two key components:
  - Integer-valued, homogeneous Lévy basis: Generates random point pattern and determines marginal distribution and cross-sectional dependence.
  - Trawl: Thins the point pattern and determines the autocorrelation structure.
- Methodology for simulation and inference for multivariate integer-valued trawl processes.
- Simulation study reveals good finite sample performance of the inference method.
- Empirical application: Joint model for the number of order submissions and cancellations in a limit order book.

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