A Malliavin-Skorohod calculus in L^0 and L^1 for pure jump additive and Volterra-type processes

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ABSTRACT

- In this paper we extend the Malliavin-Skorohod type calculus for pure jump additive processes to the L^0 and L^1 settings.
- We apply it to extend stochastic integration with respect to volatility modulated pure jump additive-driven Volterra processes.
- In particular, we define integrals with respect to Volterra processes driven by α-stable processes with $\alpha < 2$.

Motivation I

- Consider a pure jump volatility modulated additive driven Volterra ($\mathcal{VMAV}$) process X defined as

$$X(t) = \int_0^t g(t,s)\sigma(s)dJ(s)$$

provided the integral is well defined. Here J is a pure jump additive process, g is a deterministic function and σ is a predictable process with respect the natural completed filtration of J.

- This kind of models, called volatility modulated Volterra processes, are part of the family of Ambit processes and are used in modeling turbulence, energy finance and others.

Motivation II

- A major problem is to develop an integration theory with respect X as integrator, that is, to give a meaning to

$$\int_0^t Y(s)dX(s)$$

for a fixed t and a suitable stochastic processes Y. Recall that X is not necessarily a semimartingale.

- This has been done in [BBPV], assuming J is a square integrable pure jump Lévy process and assuming Malliavin regularity conditions on Y in the L^2 setting.

Motivation III

- Here we extend this integration theory to any pure jump additive process, not necessary square integrable, and in particular allowing to treat integration, for example, with respect to α-stable processes when $\alpha < 2$.
- Integrability conditions related with Y are in the L^1 setting. So, our results are an extension on the previous ones in the finite activity case and treat new cases in the infinite activity case.

INTRODUCTION I

- The Malliavin-Skorohod calculus for square integrable functionals of an additive process is today a well established topic. See for example Yablonski (2008).
- In [SUV] a new canonical space for Lévy processes is introduced and a probabilistic interpretation of Malliavin-Skorohod operators in this space is obtained.
- These operators defined in the canonical space are well defined beyond the L^2 setting.

Introduction II

- This allows to explore the development of a Malliavin-Skorohod calculus for functionals adapted to a general additive processes that belong only to L^1 or L^0.
- This is the main goal of our work, that can be seen as an extension of [SUV] using also ideas from Picard (1996).
- In particular we prove several rules of calculus and a new version of the Clark-Hausmmann-Ocone (CHO) formula in the L^1 setting.

Preliminaries and Notation I

- Let $X = \{X_t, t \geq 0\}$ be an additive process, that is, a process with independent increments, stocastically continuous, null at the origin and with càdlàg trajectories.
- Let $\mathbb{R}_0 := \mathbb{R} - \{0\}$.
- For any fixed $\epsilon > 0$, denote $S_\epsilon := \{|x| > \epsilon\} \subseteq \mathbb{R}_0$.
- Let us denote $\mathcal{B}$ and $\mathcal{B}_0$ the $\sigma-$algebras of Borel sets of $\mathbb{R}$ and $\mathbb{R}_0$ respectively.

Preliminaries and Notation II

The distribution of an additive process can be characterized by the triplet $(\Gamma_t, \sigma_t^2, \nu_t),\ t \geq 0$, where

- $\{\Gamma_t, t \geq 0\}$ is a continuous function null at the origin.
- $\{\sigma_t^2, t \geq 0\}$ is a continuous and non-decreasing function null at the origin.
- $\{\nu_t, t \geq 0\}$ is a set of Lévy measures on $\mathbb{R}$. Moreover, for any set $B \in \mathcal{B}_0$ such that $B \subseteq S_\epsilon$ for a certain $\epsilon > 0$, $\nu_\cdot(B)$ is a continuous and increasing function null at the origin.

Preliminaries and Notation III

- Let $\Theta := [0, \infty) \times \mathbb{R}$. Denote $\theta := (t, x) \in \Theta$ and $d\theta = (dt, dx)$.
- For $T \geq 0$, we introduce the measurable spaces $(\Theta_{T,\epsilon}, \mathcal{B}(\Theta_{T,\epsilon}))$ where $\Theta_{T,\epsilon} := [0, T] \times S_\epsilon$.
- Observe that $\Theta_{\infty,0} = [0, \infty) \times \mathbb{R}_0$ and that Θ can be represented as $\Theta = \Theta_{\infty,0} \cup ([0, \infty) \times \{0\})$.

Preliminaries and Notation IV

- We introduce a measure ν on $\Theta_{\infty,0}$ such that for any $B \in \mathcal{B}_0$ we have $\nu([0,t] \times B) := \nu_t(B)$. The hypotheses on ν_t guarantee that $\nu(\{t\} \times B) = 0$ for any $t \geq 0$ and for any $B \in \mathcal{B}_0$. Note that in particular, ν is $\sigma-$finite.
- Let N be the jump measure associated to X. Recall that it is a Poisson random measure on $\mathcal{B}(\Theta_{\infty,0})$ with parameter ν. Denote $\widetilde{N}(dt, dx) := N(dt, dx) - \nu(dt, dx)$.
- We can introduce also a $\sigma-$finite measure σ on $[0, \infty)$ such that $\sigma([0,t]) = \sigma_t^2$.

Preliminaries ad Notation V

According to the Lévy-Itô decomposition we can write:

$$X_t = \Gamma_t + W_t + J_t, \quad t \geq 0$$

where

- Γ is a continuous deterministic function null at the origin.
- W is a centered Gaussian process with variance process σ^2.

Preliminaries and Notation VI

- J is an additive process with triplet $(0, 0, \nu_t)$ independent of W, defined by

$$J_t = \int_{\Theta_{t,1}} xN(ds, dx) + \lim_{\epsilon \downarrow 0} \int_{\Theta_{t,\epsilon} - \Theta_{t,1}} x\widetilde{N}(ds, dx)$$

  where the convergence is *a.s.* and uniform with respect to t on every bounded interval. We call the process $J = \{J_t, t \geq 0\}$ a pure jump additive process.
- Moreover, if $\{\mathcal{F}_t^W, t \geq 0\}$ and $\{\mathcal{F}_t^J, t \geq 0\}$ are, respectively, the completed natural filtrations of W and J, then, for every $t \geq 0$, we have $\mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J$.

Preliminaries and Notation VII

- We consider on Θ the $\sigma-$finite Borel measure

$$\mu(dt, dx) := \sigma(dt)\delta_0(dx) + \nu(dt, dx).$$

  Note that μ is continuous in the sense that $\mu(\{t\} \times B) = 0$ for all $t \geq 0$ and $B \in \mathcal{B}$.

- Then we define

$$M(dt, dx) = (W \otimes \delta_0)(dt, dx) + \tilde{N}(dt, dx)$$

  that is a centered random measure with independent values such that $\mathbb{E}\big[M(E_1)M(E_2)] = \mu(E_1 \cap E_2)$, for $E_1, E_2 \in \mathcal{B}(\Theta)$ with $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$.

Preliminaries and Notation VIII

- If we take $\sigma^2 \equiv 0$, $\mu = \nu$ and $M = \tilde{N}$, we recover the Poisson random measure case.
- If we take $\nu = 0$, we have $\mu(dt, dx) := \sigma(dt)\delta_0(dx)$ and $M(ds, dx) = (W \otimes \delta_0)(ds, dx)$ and we recover the independent increment centered Gaussian measure case.
- If we take $\sigma_t^2 := \sigma_L^2 t$ and $\nu(dt, dx) = dt\nu_L(dx)$, we obtain $M(ds, dx) = \sigma_L(W \otimes \delta_0)(ds, dx) + \tilde{N}(ds, dx)$ and we recover the Lévy case (stationary increments case).

Malliavin-Skorohod calculus for additive processes in L^2.

We recall the presentation of the Malliavin-Skorohod calculus with respect to the random measure M on its canonical space in the L^2-framework, as a first step towards our final goal of extending the calculus to the L^1 and L^0 frameworks.

- Given μ, we can consider the spaces

$$\mathbb{L}_n^2 := L^2\left(\Theta^n, \mathcal{B}(\Theta)^{\otimes n}, \mu^{\otimes n}\right)$$

and define for functions f in $\mathbb{L}_n^2$ the Itô multiple stochastic integrals $I_n(f)$ with respect to M in the usual way.

- Then we have the so-called chaos representation property, that is, for any functional $F \in L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, where $\mathcal{F}^X = \vee_{t \geq 0} \mathcal{F}_t^X$, we have

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

for a certain unique family of symmetric kernels $f_n \in \mathbb{L}_n^2$.

The Malliavin and Skorohod operators I

The chaos representation property of $L^2(\Omega, \mathcal{F}^X, \mathbb{P})$ shows that this space has a Fock space structure. Thus it is possible to apply all the machinery related to the anhilation operator (Malliavin derivative) and the creation operator (Skorohod integral).

Consider $F = \sum_{n=0}^{\infty} I_n(f_n)$, with f_n symmetric and such that $\sum_{n=1}^{\infty} n\, n! \|f_n\|^2_{\mathbb{L}^2_n} < \infty$. The Malliavin derivative of F is an object of $L^2(\Theta \times \Omega, \mu \otimes \mathbb{P})$, defined as

$$D_\theta F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n(\theta, \cdot)\Big), \quad \theta \in \Theta.$$

We denote by $\mathrm{Dom} D$ the domain of this operator.

The Malliavin and Skorohod operators II

Let $u \in L^2(\Theta \times \Omega, \mathcal{B}(\Theta) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P})$. For every $\theta \in \Theta$ we have the chaos decomposition

$$u_\theta = \sum_{n=0}^{\infty} I_n(f_n(\theta, \cdot))$$

where $f_n \in \mathbb{L}^2_{n+1}$ is symmetric in the last n variables. Let $\tilde{f}_n$ be the symmetrization in all $n+1$ variables. Then we define the Skorohod integral of u by

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n),$$

in $L^2(\Omega)$, provided $u \in \operatorname{Dom}\delta$, that means $\sum_{n=0}^{\infty}(n+1)! \, \|\tilde{f}_n\|^2_{\mathbb{L}^2_{n+1}} < \infty$.

Duality between the Malliavin and Skorohod operators

- If $u \in \mathrm{Dom}\,\delta$ and $F \in \mathrm{Dom}\,D$ we have the duality relation

$$\mathbb{E}[\delta(u)\,F] = \mathbb{E}\int_{\Theta} u_\theta\, D_\theta F\, \mu(d\theta).$$

- We recall that if $u \in Dom\delta$ is actually predictable with respect to the filtration generated by X, then the Skorohod integral coincides with the (non anticipating) Itô integral in the L^2-setting with respect to M.

The Clark-Haussmann-Ocone formula I

Let $A \in \mathcal{B}(\Theta)$ and $\mathcal{F}_A := \sigma\{M(A') : A' \in \mathcal{B}(\Theta), A' \subseteq A\}$.

- F is $\mathcal{F}_A-$measurable if for any $n \geq 1$, $f_n(\theta_1, \ldots, \theta_n) = 0$, $\mu^{\otimes n} - a.e.$ unless $\theta_i \in A \quad \forall\ i = 1, \ldots n$.
- In particular, we are interested in the case $A = \Theta_{t-} := [0, t) \times \mathbb{R}$. Denote $\mathcal{F}_{t-} := \mathcal{F}_{\Theta_{t-}}$. Obviously, if $F \in \mathrm{Dom}\, D$ and it is $\mathcal{F}_{t-}-$measurable then $D_{s,x}F = 0$ for a.e. $s \geq t$ and any $x \in \mathbb{R}$.

The Clark-Haussmann-Ocone formula II

From the chaos representation property we can see that for $F \in L^2(\Omega)$,

$$E[F|\mathcal{F}_{t-}] = \sum_{n=0}^{\infty} I_n(f_n(\theta_1, \dots, \theta_n) \prod_{i=1}^{n} 1\!\!1_{[0,t)}(t_i)).$$

Then, for $F \in DomD$ we have

$$D_{s,x} E[F|\mathcal{F}_{t-}] = E[D_{s,x} F|\mathcal{F}_{t-}] 1\!\!1_{[0,t)}(s),\ (s,x) \in \Theta.$$

THE CLARK-HAUSSMANN-OCONE FORMULA III

Using these facts we can prove the very well known CHO formula:
If $F \in DomD$ we have

$$F = \mathbb{E}(F) + \delta(E[D_{t,x}F|\mathcal{F}_{t-}]).$$

- Note that being the integrand a predictable process, the Skorohod integral δ here above is actually an Itô integral.
- Note also that the CHO formula can be rewritten in a decompactified form as

$$F = \mathbb{E}(F) + \int_0^\infty E(D_{s,0}F|\mathcal{F}_{s-})dW_s + \int_{\Theta_{\infty,0}} E(D_{s,x}F|\mathcal{F}_{s-})\tilde{N}(ds, dx).$$

A CANONICAL SPACE FOR J I

- We set our work on the canonical space of J, substantially introduced in [SUV].
- The construction is done first of all in the case ν is concentrated on $\Theta_{T,\epsilon}$ for a fixed $T > 0$ and $\epsilon > 0$, that is a finite activity case. Later the construction is extended to the case $\Theta_{\infty,0}$ taking $T \uparrow \infty$ and $\epsilon \downarrow 0$.
- In the case ν concentrated on $\Theta_{T,\epsilon}$, and so finite, any trajectory of J can be totally described by a finite sequence $\big((t_1, x_1), \ldots, (t_n, x_n)\big)$ where $t_1, \ldots, t_n \in [0, T]$ are the jump instants, with $t_1 < t_2 < \cdots < t_n$, and $x_1, \ldots, x_n \in S_\epsilon$ are the corresponding sizes, for some n.

A CANONICAL SPACE FOR J II

- The extension to the space $\Theta_{\infty,0}$ is done through a projective system of probability spaces.
- For every $m \geq 1$ we consider the probability spaces

$$(\Omega_m^J, \mathcal{F}_m, \mathbb{P}_m) := (\Omega_{m,\frac{1}{m}}^J, \mathcal{F}_{m,\frac{1}{m}}, \mathbb{P}_{m,\frac{1}{m}}),$$

  that are the canonical spaces corresponding to $\Theta_m := [0, m] \times S_{\frac{1}{m}}$.
- Then the canonical space Ω^J for J on $\Theta_{\infty,0}$ is defined as the projective limit of the system $(\Omega_m^J, m \geq 1)$.

A CANONICAL SPACE FOR J III

In our setup, $\Omega^J = \cup_{n=0}^{\infty} \Theta_{\infty,0}^n$ and the probability measure $\mathbb{P}$ is concentrated on the subset of

- The empty sequence α, corresponding to the element $(\alpha, \alpha, \dots)$.
- All finite sequences of pairs (t_i, x_i).
- All infinite sequences of pairs (t_i, x_i) such that for every $m > 0$ there is only a finite number of (t_i, x_i) on Θ_m.

Malliavin-Skorohod calculus for pure jump additive processes

- Now we establish the basis for a Malliavin-Skorohod calculus with respect to a pure jump additive process, constructively on the canonical space.
- In general, the proofs of the following results are done directly on Ω_m^J and extended to Ω^J by dominated convergence.

TRANSFORMATIONS ON THE CANONICAL SPACE

- Let $\theta = (s, x) \in \Theta_{\infty,0}$. Let $\omega \in \Omega^J$, that is, $\omega := (\theta_1, \ldots, \theta_n, \ldots)$, with $\theta_i := (s_i, x_i)$.
- We introduce the following two transformations from $\Theta_{\infty,0} \times \Omega^J$ to Ω^J :

$$\epsilon_\theta^+ \omega := \big((s, x), (s_1, x_1), (s_2, x_2), \ldots\big),$$

where a jump of size x is added at time s, and

$$\epsilon_\theta^- \omega := \big((s_1, x_1), (s_2, x_2), \ldots\big) - \{(s, x)\},$$

where we take away the point $\theta = (s, x)$ from ω.

Properties of the transformations

- These two transformations are analogous to the ones introduced in Picard (1996).
- Observe that ϵ^+ is well defined except on the set $\{(\theta, \omega) : \theta \in \omega\}$ that has null measure with respect $\nu \otimes \mathbb{P}$. We can consider by convention that on this set, $\epsilon_\theta^+ \omega := \omega$.
- The case of ϵ_θ^- is also clear. In fact this operator satisfies $\epsilon_\theta^- \omega = \omega$ except on the set $\{(\theta, \omega) : \theta \in \omega\}$.
- For simplicity of the notation sometimes we will denote $\hat{\omega}_i := \epsilon_{\theta_i}^- \omega$.

The operator T I

- For a random variable $F \in L^0(\Omega^J)$, we define the operator

$$T : L^0(\Omega^J) \mapsto L^0(\Theta_{\infty,0} \times \Omega^J),$$

  such that $(T_\theta F)(\omega) := F(\epsilon_\theta^+ \omega)$.
- It is not difficult to see that if F is a $\mathcal{F}^J$-measurable, then

$$(T.F)(\cdot) \colon \Theta_{\infty,0} \times \Omega^J \longrightarrow \mathbb{R}$$

  is $\mathcal{B}(\Theta_{\infty,0}) \otimes \mathcal{F}^J-$ measurable and $F = 0$, $\mathbb{P}$-a.s. implies $T.F(\cdot) = 0$, $\nu \otimes \mathbb{P}$-a.e. So, T is a closed linear operator defined on the entire $L^0(\Omega^J)$.

THE OPERATOR T II

But if we want to assure $T.F(\cdot) \in L^1(\Theta_{\infty,0} \times \Omega^J)$ we have to restrict the domain and guarantee that

$$\mathbb{E} \int_{\Theta_{\infty,0}} |T_\theta F| \nu(d\theta) < \infty.$$

This requires a condition that is strictly stronger than $F \in L^1(\Omega^J)$.

The operator T III

Concretely, denoting $k_m := e^{-\nu(\Theta_m - \Theta_{m-1})}$, we have to assume that

$$\sum_{m=1}^{\infty} k_m \sum_{n=0}^{\infty} \frac{n}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty,$$

whereas $F \in L^1(\Omega)$ is equivalent only to

$$\sum_{m=1}^{\infty} k_m \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty.$$

THE OPERATOR S I

For a random field $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$ we define the operator

$$S : DomS \subseteq L^0(\Theta_{\infty,0} \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(Su)(\omega) := \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) := \sum_i u_{\theta_i}(\hat{\omega}_i) < \infty.$$

In particular, if $\omega = \alpha$, we define $(Su)(\alpha) = 0$.

THE OPERATOR S II

The operator S is well defined and closed from $L^1(\Theta_{\infty,0} \times \Omega^J)$ to $L^1(\Omega)$ as the following proposition says:

PROPOSITION

If $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$, Su is well defined and takes values in $L^1(\Omega)$. Moreover

$$\mathbb{E} \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) = \mathbb{E} \int_{\Theta_{\infty,0}} u_\theta(\omega) \nu(d\theta).$$

The operator S III

Given $\theta = (s, x)$ we can define for any ω, $\tilde{\omega}_s$ as the ω restricted to jump instants strictly before s. In this case, obviously, $\epsilon_\theta^- \tilde{\omega}_s = \tilde{\omega}_s$.
If u is predictable we have $u_\theta(\omega) = u_\theta(\tilde{\omega}_s)$ and so

$$u_\theta(\epsilon_\theta^- \omega) = u_\theta(\omega),$$

and

$$(Su)(\omega) = \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) = \int_{\Theta_{\infty,0}} u_\theta(\omega) N(d\theta, \omega).$$

THE ABSTRACT DUALITY RELATION

The following theorem is the fundamental relationship between operators S and T:

THEOREM

Consider $F \in L^0(\Omega^J)$ *and* $u \in DomS$. *Then* $F \cdot Su \in L^1(\Omega^J)$ *if and only if* $TF \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ *and in this case*

$$\mathbb{E}(F \cdot Su) = \mathbb{E} \int_{\Theta_{\infty,0}} T_\theta F \cdot u_\theta \, \nu(d\theta).$$

Rules of calculus

- If u and $TF \cdot u$ belong to $DomS$ we have

$$F \cdot Su = S(TF \cdot u), \, \mathbb{P} - a.e.$$

- If u and Tu are in $DomS$ then

$$T_\theta(Su) = u_\theta + S(T_\theta u), \, \nu \otimes \mathbb{P} - a.e.$$

The operator Ψ

Now we introduce the operator $\Psi_{t,x} := T_{t,x} - Id$. Observe that this operator is linear, closed and satisfies the property

$$\Psi_{t,x}(F\,G) = G\,\Psi_{t,x}F + F\,\Psi_{t,x}G + \Psi_{t,x}(F)\,\Psi_{t,x}(G).$$

THE OPERATOR $\mathcal{E}$

On other hand, for $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$ we consider the operator:

$$\mathcal{E} : Dom\mathcal{E} \subseteq L^0(\Theta_{\infty,0} \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(\mathcal{E}u)(\omega) := \int_{\Theta_{\infty,0}} u_\theta(\omega)\nu(d\theta).$$

Note that $Dom\mathcal{E}$ is the subset of processes in $L^0(\Theta_{\infty,0} \times \Omega^J)$ such that $u(\cdot,\omega) \in L^1(\Theta_{\infty,0})$, $\mathbb{P}-$a.e.
We have also that

$$\int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega)\nu(d\theta) = \int_{\Theta_{\infty,0}} u_\theta(\omega)\nu(d\theta), \mathbb{P} - a.s.$$

THE OPERATOR Φ

Then, for $u \in Dom\Phi := DomS \cap Dom\mathcal{E} \subseteq L^0(\Theta_{\infty,0} \times \Omega^J)$, we define

$$\Phi u := Su - \mathcal{E}u.$$

Note that

- $L^1(\Theta_{\infty,0} \times \Omega^J) \subseteq Dom\Phi$.
- $E(\Phi u) = 0$, for any $u \in L^1(\Theta_{\infty,0} \times \Omega)$.
- For any $u \in Dom\Phi$, predictable,

$$\Phi(u) = \int_{\Theta_{\infty,0}} u_\theta(\omega)\tilde{N}(d\theta, \omega).$$

- $u \in L^2(\Theta_{\infty,0} \times \Omega^J)$ not implies $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ nor $u \in Dom\Phi$.

Duality between Ψ and Φ

As a corollary of the duality between T and S we have the following result:

Proposition

Consider $F \in L^0(\Omega^J)$ and $u \in Dom\Phi$. Assume also $F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$. Then $F \cdot \Phi u \in L^1(\Omega^J)$ if and only if $\Psi F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case

$$\mathbb{E}(F \cdot \Phi u) = \mathbb{E}\left(\int_{\Theta_{\infty,0}} \Psi_\theta F \cdot u_\theta \, \nu(d\theta)\right).$$

Rules of calculus

- If $F \in L^0(\Omega^J)$ and u, $F \cdot u$ and $\Psi F \cdot u$ belong to $Dom\Phi$ we have

$$\Phi(F \cdot u) = F \cdot \Phi u - \Phi(\Psi F \cdot u) - \mathcal{E}(\Psi F \cdot u), \, \mathbb{P} - a.s.$$

- If u and Ψu belong to $Dom\Phi$ we have

$$\Psi_\theta(\Phi u) = u_\theta + \Phi(\Psi_\theta u), \, \nu \otimes \mathbb{P} - a.e.$$

Relationships between the intrinsic operators and the Malliavin-Skorohod operators.

Consider now the operators D and δ restricted to the pure jump case, that is associated to the measure $\tilde{N}(ds, dx)$. We write D^J and δ^J. We have the following result:

Lemma

For any n, consider the set $\Theta^{n,*}_{T,\epsilon} = \{(\theta_1, \ldots, \theta_n) \in \Theta^n_{T,\epsilon} : \theta_i \neq \theta_j \text{ if } i \neq j\}$. *Then, for any* $g_k \in L^2(\Theta^{k,*}_{\infty,0})$ *for* $k \geq 1$ *and* $\omega \in \Omega^J$ *we have, a.s.,*

$$I_k(g_k)(\omega) = \int_{\Theta^{k,*}_{T,\epsilon}} g_k(\theta_1 \ldots, \theta_k)\tilde{N}(\omega, d\theta_1) \cdots \tilde{N}(\omega, d\theta_k).$$

The proof is based on the fact that both expressions coincide for simple functions and define bounded linear operators.

Relationship between D^J, δ^J, Ψ and Φ

- For a fixed $k \geq 0$, consider $F = I_k(g_k)$ with g_k a symmetric function of $L^2(\Theta_{\infty,0}^{k,*})$. Then, F belongs to $DomD^J \cap Dom\Psi$ and

$$D^J I_k(g_k) = \Psi I_k(g_k),\ \nu \otimes \mathbb{P} - \text{a.e.}$$

- For fixed $k \geq 1$, consider $u_\theta = I_k(g_k(\cdot, \theta))$ where $g_k(\cdot, \cdot) \in L^2(\Theta_{\infty,0}^{k+1,*})$ is symmetric with respect to the first k variables. Assume also $u \in Dom\Phi$. Then,

$$\Phi(u) = \delta^J(u),\ \mathbb{P} - \text{a.e.}.$$

RELATIONSHIP BETWEEN THE OPERATORS

- Let $F \in L^2(\Omega^J)$. Then, $F \in DomD^J \iff \Psi F \in L^2(\Theta_{\infty,0} \times \Omega_J)$, and in this case

$$D^J F = \Psi F, \ \nu \otimes P - \text{a.e.}$$

- Let $u \in L^2(\Theta_{\infty,0} \times \Omega_J) \cap Dom\Phi$. Then $u \in Dom\delta^J \iff \Phi u \in L^2(\Omega^J)$, and in this case

$$\delta^J u = \Phi u, \quad \mathbb{P} - \text{a.s.}$$

As an application of the previous results in the pure jump case we hereafter prove a CHO-type formula as an integral representation of random variables in $L^1(\Omega^J)$.

Theorem

Let $F \in L^1(\Omega^J)$ and assume $\Psi F \in L^1(\Theta_{\infty,0} \times \Omega^J)$. Then

$$F = \mathbb{E}(F) + \Phi(E(\Psi_{t,x} F | \mathcal{F}_{t-})), \ a.s.$$

Remark

Observe that under the conditions of the previous theorem we have

$$\Psi_{s,x} E[F|\mathcal{F}_{\Theta_{t-}}] = E[\Psi_{s,x} F|\mathcal{F}_{\Theta_{t-}}] 1\!\!1_{[0,t)}, \; \nu \otimes \mathbb{P} - a.e.$$

Example 1 I

Consider a pure jump additive process L. On one hand, for any t, we have the Lévy-Itô decomposition:

$$L_t = \Gamma_t + \int_0^t \int_{\{|x|>1\}} xN(ds, dx) + \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds, dx).$$

Consider L_T. Assume $\mathbb{E}(|L_T|) < \infty$. Recall that this is equivalently to

$$\int_0^t \int_{|x|>1} |x|\nu(ds, dx) < \infty.$$

Then we can write

$$L_t = \Gamma_t + \int_0^t \int_{\{|x|>1\}} x\nu(ds, dx) + \int_0^t \int_{\mathbb{R}} x\tilde{N}(ds, dx).$$

EXAMPLE 1 II

On the other hand, applying the CHO formula, we have $\Psi_{s,x} L_T = x 1\!\!1_{[0,T]}(s)$ and $E(\Psi_{s,x} L_T | \mathcal{F}_{s-}) = x 1\!\!1_{[0,T)}(s)$. So, the hypothesis $\mathbb{E}(|L_T|) < \infty$ is equivalent to

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} |\Psi_{s,x} L_T| \nu(ds, dx) < \infty$$

and

$$L_T = \mathbb{E}(L_T) + \int_0^T \int_{\mathbb{R}} x \tilde{N}(ds, dx).$$

Observe that this is coherent with the previous decomposition because

$$\mathbb{E}(L_T) = \Gamma_T + \int_0^T \int_{\{|x|>1\}} x \nu(ds, dx).$$

Example 2 I

Let $X := \{X_t, t \in [0,T]\}$ be a pure jump Lévy process with triplet $(\gamma_L t, 0, \nu_L t)$. Let $S_t := e^{X_t}$ be an asset price process. Let $\mathbb{Q}$ be a risk-neutral measure. In order $e^{-rt}e^{X_t}$ be a $\mathbb{Q}-$martingale we need to assume some restrictions on ν_L and γ_L:

$$\int_{|x|\geq 1} e^x \nu_L(dx) < \infty$$

and

$$\gamma_L = \int_{\mathbb{R}} (e^y - 1 - y \mathbb{1}_{\{|y|<1\}}) \nu(dy).$$

Example 2 II

These conditions allow us to write without loosing generality,

$$X_t = x + (r - c_2)t + \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy),$$

where

$$c_2 := \int_{\mathbb{R}} (e^y - 1 - y)\nu_L(dy)$$

and N is a Poisson random measure under $\mathbb{Q}$.
According to the CHO formula if $F = S_T \in L^1(\Omega)$ and $\mathbb{E}_{\mathbb{Q}}[\Psi_{s,x} S_T | \mathcal{F}_{s-}] \in L^1(\Omega \times [0, T])$ we have

$$S_T = \mathbb{E}_{\mathbb{Q}}(S_T) + \int_{\Theta_{T,0}} \mathbb{E}_{\mathbb{Q}}[\Psi_{s,x} S_T | \mathcal{F}_{s-}] \tilde{N}(ds, dx).$$

Example 2 III

Observe that

$$\Psi_{s,x} S_T(\omega) = S_T(e^x - 1),\ \ell \times \nu_L \times \mathbb{Q} - a.s.,$$

and this process belongs to $L^1(\Omega \times \Theta_{\infty,0})$ if and only if $\int_{\mathbb{R}} |e^x - 1| \nu_L(dx) < \infty$.
Then, in this case, we have

$$S_T = \mathbb{E}_{\mathbb{Q}}(S_T) + \int_{\Theta_{T,0}} e^{r(T-s)}(e^x - 1) S_{s-} \tilde{N}(ds, dx).$$

So, this result covers Lévy processes with finite activity and Lévy processes with infinite activity but finite variation.

Integration with respect pure jump volatility modulated Volterra processes

Consider a pure jump volatility modulated additive driven Volterra ($\mathcal{VMAV}$) process X defined as

$$X(t) = \int_0^t g(t,s)\sigma(s)dJ(s)$$

provided the integral is well defined. Here J is a pure jump additive processes, g is a deterministic function and σ is a predictable process with respect the natural completed filtration of J.

Integration with respect pure jump volatility modulated Volterra processes

Recall that using the Lévy-Itô representation J can be written as

$$J(t) = \Gamma_t + \int_{\Theta_{t,0}-\Theta_{t,1}} x\tilde{N}(ds, dx) + \int_{\Theta_{t,1}} xN(ds, dx),$$

where Γ is a continuous deterministic function that we assume of bounded variation in order to admit integration with respect $d\Gamma$.

Integration with respect pure jump volatility modulated Volterra processes

For each t, X_t is well defined if

$$(H1) : \int_0^\infty |g(t,s)\sigma(s)| d\Gamma_s < \infty,$$

$$(H2) : \int_{\Theta_{\infty,0}} (1 \wedge (g(t,s)\sigma(s)x)^2)\nu(dx,ds) < \infty,$$

and

$$(H3) : \int_{\Theta_{\infty,0}} |g(t,s)\sigma(s)x[\mathbb{1}_{\{|g(t,s)\sigma(s)x|\leq 1\}} - \mathbb{1}_{\{|x|\leq 1\}}]|\nu(dx,ds) < \infty.$$

Integration with respect pure jump volatility modulated Volterra processes

Hereafter we discuss the problem of developing an integration theory with respect to X as integrator, i.e. to give a meaning to

$$\int_0^t Y(s)dX(s)$$

for a fixed t and a suitable stochastic processes Y.

Integration with respect pure jump volatility modulated Volterra processes

- Exploiting the representation of J, an integration with respect to X can be treated as the sum of integrals with respect to the corresponding components of J.
- It is enough to define integrals with respect $\int_0^t g(t,s)\sigma(s)d\Gamma_s$, $\int_0^t \int_{|x|\leq 1} g(t,s)\sigma(s)x\tilde{N}(ds,dx)$ and $\int_0^t \int_{|x|>1} g(t,s)\sigma(s)xN(ds,dx)$.
- Under the assumptions that Γ has finite variation and using the fact that N on $[0,t]\times\{|x|>\delta\}$, for any $\delta>0$, is a.s. a finite measure, the integration with respect to the first and third term presents no difficulties.

Integration with respect pure jump volatility modulated Volterra processes

- We have to discuss the second term, specifically the case when J has infinite activity and the corresponding X is not a semimartingale. In fact, if X was a semimartingale, we could perform the integration in the Itô sense.
- We can refer to [BBPV] for a discussion of the the conditions on g in order X be or not a semimartingale
- In [BBPV], an integral with respect to a non semimartingale X driven by a Lévy process by means of the Malliavin-Skorohod calculus is defined. Their technique is naturally constrained to an L^2 setting.

Integration with respect pure jump volatility modulated Volterra processes

Within the framework presented in this paper, we can extend the definition proposed in [BBPV] to reach out for additive processes beyond the L^2 setting.
Assume the following hypothesis on X and Y:

- For $s \geq 0$, the mapping $t \longrightarrow g(t,s)$ is of bounded variation on any interval $[u,v] \subseteq (s,\infty)$.
- The function

$$\mathcal{K}_g(Y)(t,s) := Y(s)g(t,s) + \int_s^t (Y(u) - Y(s))g(du,s), \quad t > s,$$

is well defined a.s., in the sense that $(Y(u) - Y(s))$ is integrable with respect to $g(du,s)$ as a pathwise Lebesgue-Stieltjes integral.

Integration with respect pure jump volatility modulated Volterra processes

- The mappings

$$(s,x) \longrightarrow \mathcal{K}_g(Y)(t,s)\sigma(s)x\mathbb{1}_{\Theta_{t,0}-\Theta_{t,1}}(s,x)$$

and

$$(s,x) \longrightarrow \Psi_{s,x}(\mathcal{K}_g(Y)(t,s)\sigma(s))x\mathbb{1}_{\Theta_{t,0}-\Theta_{t,1}}(s,x)$$

belong to $Dom\Phi$.

Then, the following integral, is well defined:

$$\begin{aligned} & \int_0^t Y(s) d\left(\int_0^s \int_{|x| \leq 1} g(s,u)\sigma(u) x \tilde{N}(du,dx)\right) \\ := \quad & \Phi(x\mathcal{K}_g(Y)(t,s)\sigma(s) 1\!\!1_{\Theta_{t,0}-\Theta_{t,1}}(s,x)) \\ + \quad & \Phi(x\Psi_{s,x}(\mathcal{K}_g(Y)(t,s))\sigma(s) 1\!\!1_{\Theta_{t,0}-\Theta_{t,1}}(s,x)) \\ + \quad & \mathcal{E}(x\Psi_{s,x}(\mathcal{K}_g(Y)(t,s))\sigma(s) 1\!\!1_{\Theta_{t,0}-\Theta_{t,1}}(s,x)). \end{aligned}$$

Integration with respect pure jump volatility modulated Volterra processes

- This result extends the definition in [BBPV] to any pure jump additive process J, i.e. beyond square integrability.
- The proof relies on the definitions of Φ, Ψ and the developed calculus rules.
- In the finite activity case,

$$L^2(\Theta_{\infty,0} \times \Omega^J) \subseteq L^1(\Theta_{\infty,0} \times \Omega^J)$$

  and our result is an extension of the definition in [BBPV].
- In the infinite activity case, our Theorem covers cases not covered by [BBPV] and viceversa.

EXAMPLE I

- Hereafter we give a classical example of a pure jump Lévy process without second moment as a driver and we consider a kernel function g of shift type, i.e. it only depends on the difference $t - s$. For simplicity we assume moreover $\sigma \equiv 1$. The chosen kernel appears in applications to turbulence.
- Assume L to be a symmetric $\alpha-$stable Lévy process, for $\alpha \in (0, 2)$. It corresponds to the triplet $(0, 0, \nu_L)$ with $\nu_L(dx) = c|x|^{-1-\alpha}dx$.

Example II

Take

$$g(t,s) := (t-s)^{\beta-1} e^{-\lambda(t-s)} 1\!\!1_{[0,t)}(s)$$

with $\beta \in (0,1)$ and $\lambda > 0$. Note that

$$g(du,s) = -g(u,s)\left(\frac{1-\beta}{u-s} + \lambda\right)du.$$

Example III

We concentrate on the component

$$J(t) = \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds, dx),$$

and so first of all on the definition of the integral

$$X(t) := \int_0^t g(t,s)dJ(s) = \int_0^t \int_{|x|\leq 1} g(t,s)x\tilde{N}(ds,dx),\ t \geq 0.$$

Example IV

In relation with this integral, that is not a semimartingale, we have four situations:

1. If $\alpha \in (0,1)$ and $\beta > \frac{1}{2}$, $g(t,s)x$ belongs to $L^1 \cap L^2$
2. If $\alpha \in (0,1)$ and $\beta \leq \frac{1}{2}$, $g(t,s)x$ belongs to L^1 but not to L^2.
3. If $\alpha \in [1,2)$ and $\beta > \frac{1}{2}$, $g(t,s)x$ belongs to L^2 but not to L^1.
4. If $\alpha \in [1,2)$ and $\beta \leq \frac{1}{2}$, $g(t,s)x$ belongs not to L^2 nor to L^1.

Only in case (4) the integral is not necessarily well defined.

Example V

Just to show the types of computation involved, let us consider the particular case of a $\mathcal{VMAV}$ process as integrand. Namely,

$$Y(s) = \int_0^s \int_{|x| \leq 1} \phi(s-u) x \tilde{N}(du, dx), \quad 0 \leq s \leq T,$$

where ϕ is a positive continuous function such that the integral Y is well defined.
Consider the case $\alpha < 1$ and $\beta \in (0, 1)$, not covered by [BBPV] if $\beta \leq \frac{1}{2}$.

Example VI

In order to see that $\int_0^t Y(s-)dX(s)$ is well defined we have to check:

1. The process $(Y(u) - Y(s))$ is integrable with respect to $g(du, s)$ on $(s, t]$, as a pathwise Lebesgue-Stieltjes integral.
2. The mappings

$$(s, x) \longrightarrow x\mathcal{K}_g(Y)(t, s)\mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{|x|\leq 1\}}$$

and

$$(s, x) \longrightarrow x\Psi_{s,x}(\mathcal{K}_g(Y)(t, s))\mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{|x|\leq 1\}}$$

belong to $Dom\Phi$.

Example VII

We have

$$
\begin{aligned}
& \mathcal{K}_g(Y)(t,s) \\
= \; & g(t,s) \int_{[0,s)} \int_{|x|\leq 1} \phi(s-v) x \tilde{N}(dv,dx) \\
- \; & \int_s^t g(u,s)\left(\frac{1-\beta}{u-s} + \lambda\right) \int_{[s,u)} \int_{|x|\leq 1} \phi(u-v) x \tilde{N}(dv,dx) du \\
- \; & \int_s^t g(u,s)\left(\frac{1-\beta}{u-s} + \lambda\right) \int_{[0,s)} \int_{|x|\leq 1} [\phi(u-v) - \phi(s-v)] x \tilde{N}(dv,dx) du
\end{aligned}
$$

Example VIII

In terms of Φ we can rewrite

$$\begin{aligned}
& \mathcal{K}_g(Y)(t,s) \\
= \quad & g(t,s)\Phi(\phi(s-\cdot)x 1\!\!1_{\{|x|\leq 1\}} 1\!\!1_{[0,s)}) \\
- \quad & \int_s^t g(u,s)(\frac{1-\beta}{u-s}+\lambda)\Phi(\phi(u-\cdot)x 1\!\!1_{\{|x|\leq 1\}} 1\!\!1_{[s,u)}(\cdot))du \\
- \quad & \int_s^t g(u,s)(\frac{1-\beta}{u-s}+\lambda)\Phi([\phi(u-\cdot)-\phi(s-\cdot)]x 1\!\!1_{\{|x|\leq 1\}} 1\!\!1_{[0,s)}(\cdot))du.
\end{aligned}$$

Moreover we have

$$\Psi_{s,x}\mathcal{K}_g(Y)(t,s) = -x 1\!\!1_{\{|x|\leq 1\}} \int_s^t g(u,s)\phi(u-s)(\frac{1-\beta}{u-s}+\lambda) 1\!\!1_{[0,u)}(s)du.$$

Example IX

So, it is enough to check that the mappings

$$(s,x) \longrightarrow x\mathcal{K}_g(Y)(t,s)\mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{|x|\leq 1\}}$$

and

$$(s,x) \longrightarrow x\Psi_{s,x}(\mathcal{K}_g(Y)(t,s))\mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{|x|\leq 1\}}$$

are in $L^1(\Theta_{\infty,0} \times \Omega)$.

Example X

If for example we consider the case $\phi(y) = y^\gamma$ with $\gamma > 0$ and $\beta + \gamma \geq 1$ is not difficult to check the mappings are in L^1 and we conclude that the integral

$$\int_0^t Y(s-)dX(s)$$

is well defined.

Thank you for the attention

Tak

Gràcies