

Rough paths methods 2: Young integration

Samy Tindel

Purdue University

University of Aarhus 2016

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration
- 3 Increments
- 4 Algebraic Young integration
- 5 Differential equations

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration
- 3 Increments
- 4 Algebraic Young integration
- 5 Differential equations

Definition of fBm

Complete probability space: $(\Omega, \mathcal{F}, \mathbf{P})$

Definition 1.

A 1-d fBm is a continuous process $B = \{B_t; t \geq 0\}$ such that:

- $B_0 = 0$
- B is a centered Gaussian process
- $\mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H})$, for $H \in (0, 1)$

d -dimensional fBm: $B = (B^1, \dots, B^d)$, with B^i independent 1-d fBm

fBm: variance of the increments

Notation: If $f : [0, T] \rightarrow \mathbb{R}^d$ is a function, we shall denote:

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \|f\|_\mu = \sup_{s,t \in [0, T]} \frac{|\delta f_{st}|}{|t - s|^\mu}$$

Variance of the increments: for a 1-d fBm,

$$\mathbf{E}[|\delta B_{st}|^2] \equiv \mathbf{E}[|B_t - B_s|^2] = |t - s|^{2H}$$

FBm regularity

Proposition 2.

FBm $B \equiv B^H$ is γ -Hölder continuous on $[0, T]$ for all $\gamma < H$, up to modification.

Proof: We have $\delta B_{st} \sim \mathcal{N}(0, |t - s|^{2H})$. Thus for $n \geq 1$,

$$\mathbf{E} \left[|\delta B_{st}|^{2n} \right] = c_n |t - s|^{2Hn} \quad \text{i.e.} \quad \mathbf{E} \left[|\delta B_{st}|^{2n} \right] = c_n |t - s|^{1+(2Hn-1)}$$

Kolmogorov: B is γ -Hölder for $\gamma < (2Hn - 1)/2n = H - 1/(2n)$.

Proof finished by letting $n \rightarrow \infty$.

Some properties of fBm

Proposition 3.

Let B be a fBm with parameter H . Then:

- 1 $\{ a^{-H} B_{at}; t \geq 0 \}$ is a fBm (scaling)
- 2 $\{ B_{t+h} - B_h; t \geq 0 \}$ is a fBm (stationarity of increments)
- 3 B is not a semi-martingale unless $H = 1/2$

Proof of claim 3

Semi-martingale and quadratic variation:

If B were a semi-martingale, we would get on $[0, 1]$:

$$\mathbf{P} - \lim_{n \rightarrow \infty} \sum_{i=1}^n (B_{i/n} - B_{(i-1)/n})^2 = \langle B \rangle_1,$$

where $\langle B \rangle$ is the (non trivial) quadratic variation of B .

We will show that $\langle B \rangle$ is trivial (0 or ∞) whenever $H \neq 1/2$.

Proof of claim 3 (2)

A p -variation: Define

$$V_{n,p} = \sum_{i=1}^n |B_{i/n} - B_{(i-1)/n}|^p, \quad \text{and} \quad Y_{n,p} = n^{pH-1} V_{n,p}.$$

By scaling properties, we have:

$$Y_{n,p} \stackrel{(d)}{=} \hat{Y}_{n,p}, \quad \text{with} \quad \hat{Y}_{n,p} = n^{-1} \sum_{i=1}^n |B_i - B_{i-1}|^p.$$

The sequence $\{B_i - B_{i-1}; i \geq 1\}$ is stationary and mixing
 $\Rightarrow \hat{Y}_{n,p}$ converges \mathbf{P} -a.s and in L^1 towards $\mathbf{E}[|B_1 - B_0|^p]$
 $\Rightarrow \mathbf{P} - \lim_{n \rightarrow \infty} Y_{n,p} = E[|B_1|^p]$
 $\Rightarrow \mathbf{P} - \lim_{n \rightarrow \infty} V_{n,p} = 0$ if $pH > 1$, ∞ if $pH < 1$

Proof of claim 3 (3)

Recall: $V_{n,p} = \sum_{i=1}^n |B_{i/n} - B_{(i-1)/n}|^p$

Definition: $\mathbf{P} - \lim_{n \rightarrow \infty} V_{n,p}^{1/p} \equiv \mathcal{V}_p(B)$ is called p -variation of B
 \Rightarrow We have seen $\mathcal{V}_p(B) = 0$ if $pH > 1$, ∞ if $pH < 1$

Property: if $p_1 < p_2$, then $\mathcal{V}_{p_1}(B) \geq \mathcal{V}_{p_2}(B)$

Case $H > 1/2$: choose $p < 2$ such that $pH > 1$
 $\Rightarrow \mathcal{V}_p(B) = 0 \Rightarrow \mathcal{V}_2(B) = 0$

Case $H < 1/2$: choose $p > 2$ such that $pH < 1$
 $\Rightarrow \mathcal{V}_p(B) = \infty \Rightarrow \mathcal{V}_2(B) = \infty$

Conclusion: if $H \neq 1/2$, Itô's type methods do not apply in order to define stochastic integrals

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration**
- 3 Increments
- 4 Algebraic Young integration
- 5 Differential equations

Strategy for $H > 1/2$

- Generally speaking, take advantage of two aspects of fBm:
 - ▶ Gaussianity
 - ▶ Regularity

For $H > 1/2$, regularity is almost sufficient

- Notation: $\mathcal{C}_1^\gamma = \mathcal{C}_1^\gamma(\mathbb{R}) \equiv \gamma$ -Hölder functions of 1 variable
- If $H > 1/2$, $B \in \mathcal{C}_1^\gamma$ for any $1/2 < \gamma < H$ a.s
- We shall try to solve our equation in a pathwise manner

Equation under consideration

$$X_t = a + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \quad t \in [0, T] \quad (1)$$

- $a \in \mathbb{R}^n$ initial condition
- b, σ coefficients in C_b^1
- $B = (B^1, \dots, B^d)$ d -dimensional Brownian motion
- B^i iid Brownian motions

Notational simplification

Simplified setting: In order to ease notations, we shall consider:

- Real-valued solution and fBm: $n = d = 1$.
However, we shall use d -dimensional methods
- $b \equiv 0$

Simplified equation: we end up with

$$X_t = a + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T] \quad (2)$$

- $a \in \mathbb{R}$, $\sigma \in C_b^1(\mathbb{R})$
- B is a 1-d Brownian motion

Pathwise strategy

Aim: Let x be a function in \mathcal{C}_1^γ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s) dx_s \quad (3)$$

Steps:

- Define an integral $\int z_s dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (3) through fixed point argument in \mathcal{C}_1^κ with $1/2 < \kappa < \gamma$

Notation: We set

$$\mathcal{I}_{st}(z dx) = \int_s^t z_w dx_w$$

for reasonable extensions of Riemann's integral

Particular Riemann sums

Aim: Define $\int_0^1 z_s dx_s$ for $z \in \mathcal{C}_1^\kappa, x \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$

Dyadic partition: set $t_i^n = i/2^n$, for $n \geq 0, 0 \leq i \leq 2^n$

Associated Riemann sum:

$$I_n \equiv \sum_{i=0}^{2^n-1} z_{t_i^n} [x_{t_{i+1}^n} - x_{t_i^n}] = \sum_{i=0}^{2^n-1} z_{t_i^n} \delta x_{t_i^n t_{i+1}^n}.$$

Question: Can we define $\mathcal{J}_{01}(z dx) \equiv \lim_{n \rightarrow \infty} I_n$?

Possibility: Control $|I_{n+1} - I_n|$ and write (if the series is convergent):

$$\mathcal{J}_{01}(z dx) = I_0 + \sum_{n=0}^{\infty} (I_{n+1} - I_n).$$

Control of $I_{n+1} - I_n$

We have:

$$I_n = \sum_{i=0}^{2^n-1} z_{t_i^n} \delta x_{t_i^n t_{i+1}^n} = \sum_{i=0}^{2^n-1} z_{t_{2i}^{n+1}} \left[\delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right]$$

$$I_{n+1} = \sum_{i=0}^{2^{n+1}-1} \left[z_{t_{2i}^{n+1}} \delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + z_{t_{2i+1}^{n+1}} \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right]$$

Therefore:

$$\begin{aligned} |I_{n+1} - I_n| &= \left| \sum_{i=0}^{2^n-1} \delta z_{t_{2i}^{n+1} t_{2i+1}^{n+1}} \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right| \\ &\leq \sum_{i=0}^{2^n-1} \|z\|_{\kappa} |t_{2i+1}^{n+1} - t_{2i}^{n+1}|^{\kappa} \|x\|_{\gamma} |t_{2i+2}^{n+1} - t_{2i+1}^{n+1}|^{\gamma} \\ &= \frac{\|z\|_{\kappa} \|x\|_{\gamma}}{2^{\kappa+\gamma} 2^{n(\kappa+\gamma-1)}} \end{aligned}$$

Definition of the integral

We have seen: for $\alpha \equiv \kappa + \gamma - 1 > 0$ and $n \geq 0$:

$$|I_{n+1} - I_n| \leq \frac{C_{x,z}}{2^{\alpha n}}$$

Series convergence:

Obviously, $\sum_{n=0}^{\infty} (I_{n+1} - I_n)$ is a convergent series

\hookrightarrow yields definition of $\mathcal{J}_{01}(z dx)$, and more generally: $\mathcal{J}_{st}(z dx)$

Remark:

One should consider more general partitions π , with $|\pi| \rightarrow 0$

\hookrightarrow C.f Lejay (Séminaire 37)

Young integral, version 1

Proposition 4.

Let $z \in \mathcal{C}_1^\kappa([0, T])$, $x \in \mathcal{C}_1^\gamma([0, T])$, with $\kappa + \gamma > 1$, and $0 \leq s < t \leq T$. Let

- $(\pi_n)_{n \geq 0}$ a sequence of partitions of $[s, t]$ such that $\lim_{n \rightarrow \infty} |\pi_n| = 0$
- I_n corresponding Riemann sums

Then:

- 1 I_n converges to an element $\mathcal{J}_{st}(z dx)$
- 2 The limit does not depend on the sequence $(\pi^n)_{n \geq 0}$
- 3 Integral linear in z , and coincides with Riemann's integral for smooth z, x
- 4 If $0 \leq s < u < t \leq T$, we have $\mathcal{J}_{st}(z dx) = \mathcal{J}_{su}(z dx) + \mathcal{J}_{ut}(z dx)$

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration
- 3 Increments**
- 4 Algebraic Young integration
- 5 Differential equations

Notations: increments

Simplex: For $k \geq 2$ and $T > 0$ we set

$$\mathcal{S}_{k,T} = \{(s_1, \dots, s_k); 0 \leq s_1 < \dots < s_k \leq T\}$$

$(k - 1)$ -increment: Let $T > 0$, a vector space V and $k \geq 1$:

$$\mathcal{C}_k(V) \equiv \left\{ g \in C(\mathcal{S}_{k,T}; V); \lim_{t_i \rightarrow t_{i+1}} g_{t_1 \dots t_k} = 0, i \leq k - 1 \right\}$$

Remark: We mostly consider $V = \mathbb{R}$ for notational sake

\hookrightarrow We write $\mathcal{C}_k = \mathcal{C}_k([0, T]; \mathbb{R})$

Notations: operator δ

Operator δ :

$$\delta : \mathcal{C}_k \rightarrow \mathcal{C}_{k+1}, \quad \delta g_{t_1 \dots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{\hat{\mathbf{t}}_i^{k+1}},$$

where

$$\hat{\mathbf{t}}^{k+1} = (t_1, \dots, t_{k+1})$$

$$\hat{\mathbf{t}}_i^{k+1} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{k+1})$$

Examples: if $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$ we have, for $s, u, t \in \mathcal{S}_{3,T}$,

$$\delta g_{st} = g_t - g_s, \quad \text{and} \quad \delta h_{sut} = h_{st} - h_{su} - h_{ut}.$$

First properties of δ

Proposition 5.

$\delta\delta : \mathcal{C}_k \rightarrow \mathcal{C}_{k+2}$ satisfies $\delta\delta = 0$

Notation: $\mathcal{ZC}_k = [\mathcal{C}_k \cap \text{Ker}\delta]$

Proposition 6.

Let

- $k \geq 1$
- $h \in \mathcal{ZC}_{k+1}$

There exists a (non unique) $f \in \mathcal{C}_k$ such that $h = \delta f$.

Proofs

Proposition 5, easy case: If $k = 1$, $g \in \mathcal{C}_1$ and $h \equiv \delta g$, then:

$$\begin{aligned}(\delta\delta g)_{sut} &= \delta h_{sut} = h_{st} - h_{su} - h_{ut} \\ &= [g_t - g_s] - [g_u - g_s] - [g_t - g_u] \\ &= 0\end{aligned}$$

Proofs (2)

Proposition 5, general case: Let $g \in \mathcal{C}_k$. Then:

$$\begin{aligned}(\delta\delta g)_{\mathbf{t}^{k+2}} &= \sum_{i=1}^{k+2} (-1)^{k+1-i} \delta g_{\hat{\mathbf{t}}_i^{k+2}} \\ &= \sum_{i=1}^{k+1} (-1)^{k+1-i} \delta g_{\hat{\mathbf{t}}_i^{k+2}} - \delta g_{\mathbf{t}^{k+1}}\end{aligned}\tag{4}$$

Decomposition for $\hat{\mathbf{t}}_i^{k+2}$: Write $\hat{\mathbf{t}}_i^{k+2} = \mathbf{s}^{k+1}$. Then

$$s_j = t_j \quad \text{if } j \leq i-1, \quad \text{and} \quad s_j = t_{j+1} \quad \text{if } j \geq i.$$

Proofs (3)

Computation of $\delta g_{\hat{\mathbf{t}}_i}^{k+2}$: For $i \leq k+1$ we have

$$\begin{aligned}\delta g_{\hat{\mathbf{t}}_i}^{k+2} &= \delta g_{\mathbf{s}^{k+1}} = \sum_{j=1}^{k+1} (-1)^{k-j} g_{\hat{\mathbf{s}}_j}^{k+1} \\ &= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i}^{k+1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{i,j+1}^{k+2}} \\ &= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+2} (-1)^{k-j+1} g_{\hat{\mathbf{t}}_{i,j}^{k+2}} \\ &= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+1} (-1)^{k-j+1} g_{\hat{\mathbf{t}}_{i,j}^{k+2}} - g_{\hat{\mathbf{t}}_i}^{k+1} \quad (5)\end{aligned}$$

Proofs (4)

Conclusion for Proposition 5: Plugging (5) into (4), we get

$$\begin{aligned}(\delta\delta g)_{\mathbf{t}^{k+2}} &= \sum_{i=1}^{k+1} (-1)^{k+1-i} \left[\sum_{j=1}^{i-1} (-1)^{k-j} g_{\mathbf{t}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+1} (-1)^{k-j+1} g_{\mathbf{t}_{i,j}^{k+2}} \right] \\ &\quad + \sum_{i=1}^{k+1} (-1)^{k-i} g_{\mathbf{t}_i^{k+1}} - \delta g_{\mathbf{t}^{k+1}} \\ &= \sum_{1 \leq j < i \leq k+1} (-1)^{i+j-1} g_{\mathbf{t}_{j,i}^{k+2}} + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} g_{\mathbf{t}_{i,j}^{k+2}} \\ &\quad + \delta g_{\mathbf{t}^{k+1}} - \delta g_{\mathbf{t}^{k+1}} \\ &= 0\end{aligned}$$

Proofs (5)

Proposition 6, strategy: We show that the following works:

$$f_{t_1 \dots t_k} = -h_{t_1 \dots t_k} \mathcal{T}$$

Relation $\delta h = 0$: can be written as

$$\delta h_{\mathbf{t}^{k+2}} = \sum_{i=1}^{k+1} (-1)^{k+1-i} h_{\hat{\mathbf{t}}_i^{k+2}} - h_{\mathbf{t}^{k+1}} = 0 \quad (6)$$

Verification of our claim: Set $g_{t_1 \dots t_k} = h_{t_1 \dots t_k} \mathcal{T} = -f_{t_1 \dots t_k}$. Then

$$\begin{aligned} \delta g_{\mathbf{t}^{k+1}} &= \sum_{i=1}^{k+1} (-1)^{k-i} g_{\hat{\mathbf{t}}_i^{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} h_{\hat{\mathbf{t}}_i^{k+1}} \mathcal{T} \\ &= - \sum_{i=1}^{k+1} (-1)^{k+1-i} h_{\hat{\mathbf{t}}_i^{k+1}} \mathcal{T} \stackrel{(6)}{=} -h_{\mathbf{t}^{k+1}} \end{aligned}$$

Particular case of Proposition 6

Proposition 7.

Let

- $h \in \mathcal{ZC}_2$

Then

- There exists $f \in \mathcal{C}_1$ such that $h = \delta f$.
- f is unique up to a constant

Proof of existence:

Take $f_s = -h_{sT}$ as in the general case.

Proof of uniqueness:

A function f defined by its increments is unique up to a constant.

First relation with integrals

Proposition 8.

Let f and g two smooth functions on $[0, T]$. Define $I \in \mathcal{C}_2$ by

$$I_{st} = \int_s^t \left(\int_s^v df_w \right) dg_v, \quad \text{for } s, t \in [0, T].$$

Then we have, for $s < u < t$:

$$\delta I_{sut} = [f_u - f_s][g_t - g_u] = \delta f_{su} \delta g_{ut}.$$

Remark: This elementary property is important:

- δ transforms integrals into products of increments.
- We have already seen that products of the type $\delta f \delta g$
 \hookrightarrow both regularities of f and g can be used.

Proof

Invoking the very definition of δ and I :

$$\begin{aligned}(\delta I)_{sut} &= I_{st} - I_{su} - I_{ut} \\&= \int_s^t \left(\int_s^v df_w \right) dg_v - \int_s^u \left(\int_s^v df_w \right) dg_v - \int_u^t \left(\int_u^v df_w \right) dg_v \\&= \int_u^t \left(\int_s^v df_w \right) dg_v - \int_u^t \left(\int_u^v df_w \right) dg_v \\&= \int_u^t \left(\int_s^u df_w \right) dg_v \\&= \left(\int_s^u df_w \right) \left(\int_u^t dg_v \right) = \delta f_{su} \delta g_{ut}\end{aligned}$$

Hölder spaces

Aim: take into account some regularities in \mathcal{C}_k .

Case $k=2$: if $f \in \mathcal{C}_2$, set

$$\|f\|_\mu = \sup_{(s,t) \in \mathcal{S}_{2,T}} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu = \{f \in \mathcal{C}_2; \|f\|_\mu < \infty\}.$$

Case $k=1$: if $g \in \mathcal{C}_1$, set

$$\|g\|_\mu = \|\delta g\|_\mu, \quad \text{and} \quad \mathcal{C}_1^\mu = \{g \in \mathcal{C}_1; \|g\|_\mu < \infty\}.$$

Remark: $\|\cdot\|_\mu$ defines a semi-norm in \mathcal{C}_1^μ . It is a norm on

$$\mathcal{C}_{1,a}^\mu = \{g : [0, T] \rightarrow \mathbb{R}; g_0 = a, \|g\|_\mu < \infty\}$$

Hölder spaces (2)

Case $k=3$: if $h \in \mathcal{C}_3$, set

$$\|h\|_\mu = \sup_{(s,u,t) \in \mathcal{S}_{3,T}} \frac{|h_{sut}|}{|t-s|^\mu}$$

and

$$\mathcal{C}_3^\mu = \{h \in \mathcal{C}_3; \|h\|_\mu < \infty\}.$$

Operator Λ (Sewing map)

Theorem 9.

Let $\mu > 1$. There exists a unique linear application $\Lambda : \mathcal{ZC}_3^\mu \rightarrow \mathcal{C}_2^\mu$ such that

$$\delta\Lambda = \text{Id}_{\mathcal{ZC}_3^\mu} \quad \text{and} \quad \Lambda\delta = \text{Id}_{\mathcal{C}_2^\mu}.$$

Equivalent statement: for any $h \in \mathcal{C}_3^\mu$ such that $\delta h = 0$, there exists a unique element $g = \Lambda(h) \in \mathcal{C}_2^\mu$ such that $\delta g = h$.

Furthermore, for any $\mu > 1$, the application Λ is continuous from \mathcal{ZC}_3^μ to \mathcal{C}_2^μ , and

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2} \|h\|_\mu, \quad h \in \mathcal{ZC}_3^\mu.$$

Second relation with integrals

Proposition 10.

Let $g \in \mathcal{C}_2$, such that $\delta g \in \mathcal{C}_3^\mu$ with $\mu > 1$. Define

$$k = (\text{Id} - \Lambda\delta)g$$

Then

$$k_{st} = \lim_{|\Pi_{st}| \rightarrow 0} \sum_{i=0}^n g_{t_i t_{i+1}},$$

as $|\Pi_{st}| \rightarrow 0$, where Π_{st} is a partition of $[s, t]$.

Interpretation: Increment k can be seen as an integral of g .

Proof of Proposition 10

An equation for g : Thanks to Proposition 7, we have

$$k = (\text{Id} - \Lambda\delta)g \implies \delta k = 0 \implies k = \delta f$$

for $f \in \mathcal{C}_1$ unique up to a constant. Thus:

$$g = \delta f + \Lambda\delta g \tag{7}$$

Conclusion: Thanks to (7) we have

$$S_{\Pi} = \sum_{i=0}^n g_{t_i t_{i+1}} = \sum_{i=0}^n \delta f_{t_i t_{i+1}} + \sum_{i=0}^n (\Lambda\delta g)_{t_i t_{i+1}} = \delta f_{st} + \sum_{i=0}^n (\Lambda\delta g)_{t_i t_{i+1}}.$$

Then the last sum converges to zero, since $\Lambda\delta g \in \mathcal{C}_3^{1+}(V)$

Young integral: strategy

Smooth case: Let $f, g \in \mathcal{C}_1^1$. Define $I \in \mathcal{C}_2$ by

$$I_{st} = \int_s^t \left(\int_s^v df_w \right) dg_v, \quad \text{for } s, t \in [0, T].$$

Decomposition-recomposition scheme: we have

$$I = \int df \int dg \xrightarrow{\delta} (\delta f)(\delta g) \xrightarrow{\wedge} I = \int df \int dg.$$

Indeed:

- First step: already established.
- Second step: $(\delta f)(\delta g) \in \mathcal{ZC}_3^\mu$ with $\mu > 1 \implies$ Proposition 9

Important: Second step can be extended to more irregular situations
 $\hookrightarrow f \in \mathcal{C}_1^\gamma, g \in \mathcal{C}_1^\kappa$ with $\mu = \gamma + \kappa > 1$.

Operator Λ (repeated)

Theorem 11.

Let $\mu > 1$. There exists a unique linear application $\Lambda : \mathcal{ZC}_3^\mu \rightarrow \mathcal{C}_2^\mu$ such that

$$\delta\Lambda = \text{Id}_{\mathcal{ZC}_3^\mu} \quad \text{and} \quad \Lambda\delta = \text{Id}_{\mathcal{C}_2^\mu}.$$

Equivalent statement: for any $h \in \mathcal{C}_3^\mu$ such that $\delta h = 0$, there exists a unique element $g = \Lambda(h) \in \mathcal{C}_2^\mu$ such that $\delta g = h$.

Furthermore, for any $\mu > 1$, the application Λ is continuous from \mathcal{ZC}_3^μ to \mathcal{C}_2^μ , and

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2} \|h\|_\mu, \quad h \in \mathcal{ZC}_3^\mu.$$

Operator Λ : uniqueness

Definition of 2 increments:

Let M, \hat{M} be two elements in \mathcal{C}_2^μ such that $\delta M = \delta \hat{M} = h$.

Define $Q = M - \hat{M}$.

Then $\delta Q = 0$ and $Q \in \mathcal{C}_2^\mu$.

Contradiction:

Hence there exists an element $q \in \mathcal{C}_1$ such that $Q = \delta q$, and

$$|q_t - q_s| = |Q_{st}| \leq c|t - s|^\mu$$

Since $\mu > 1$, q is constant in $[0, T]$, and thus $Q = 0$.

Operator Λ : existence

Algebraic increment:

$\delta h = 0 \Rightarrow$ existence of $B \in \mathcal{C}_2$ such that $\delta B = h$.

Construction of a sequence:

Called M_{st}^n , defined for $s, t \in [0, T]$, with $s < t$

For $n \geq 0$, consider partition $\{r_i^n; i \leq 2^n\}$ of $[s, t]$, where

$$r_i^n = s + \frac{(t-s)i}{2^n}, \quad \text{for } 0 \leq i \leq 2^n.$$

For $n \geq 0$, define

$$M_{st}^n = B_{st} - \sum_{l=0}^{2^n-1} B_{r_l^n, r_{l+1}^n}.$$

Easy step: check $M_{st}^0 = 0$.

Operator Λ : existence (2)

Control of $M^n - M^{n+1}$: we have

$$\begin{aligned}M_{st}^n - M_{st}^{n+1} &= - \sum_{i=0}^{2^n-1} \left(B_{r_{2i}^{n+1}, r_{2i+2}^{n+1}} - B_{r_{2i}^{n+1}, r_{2i+1}^{n+1}} - B_{r_{2i+1}^{n+1}, r_{2i+2}^{n+1}} \right) \\ &= - \sum_{i=0}^{2^n-1} \delta B_{r_{2i}^{n+1}, r_{2i+1}^{n+1}, r_{2i+2}^{n+1}} = - \sum_{i=0}^{2^n-1} h_{r_{2i}^{n+1}, r_{2i+1}^{n+1}, r_{2i+2}^{n+1}},\end{aligned}$$

Since $h \in \mathcal{C}_3^\mu$ with $\mu > 1$, we get

$$\left| M_{st}^n - M_{st}^{n+1} \right| \leq \frac{\|h\|_\mu (t-s)^\mu}{2^\mu 2^{n(\mu-1)}},$$

Taking limits: we obtain existence of $M_{st} \equiv \lim_{n \rightarrow \infty} M_{st}^n$, such that

$$|M_{st}| \leq \frac{\|h\|_\mu}{2^\mu - 2} |t - s|^\mu.$$

Operator Λ : existence (3)

More general sequences: Consider

- $\{\pi_n; n \geq 1\}$ sequence of partitions of $[s, t]$
- $\pi_n = \{r_0^n, r_1^n, \dots, r_{k_n}^n, r_{k_n}^n\}$
- $\pi_n \subset \pi_{n+1}$, and $\lim_{n \rightarrow \infty} k_n = \infty$
- $M_{ts}^{\pi_n} = B_{ts} - \sum_{l=0}^{k_n} B_{r_{l+1}^n, r_l^n}$

Removing points of a partition:

For $n \geq 1$, there exists $1 \leq l \leq k_n$ such that

$$|r_{l+1}^n - r_{l-1}^n| \leq \frac{2|t-s|}{k_n} \quad (8)$$

Then

- Pick now such an index l
- Transform π^n into $\hat{\pi}$, where

$$\hat{\pi} = \left\{ r_0^n, r_1^n, \dots, r_{l-1}^n, r_{l+1}^n, \dots, r_{k_n}^n, r_{k_n+1}^n \right\}.$$

Operator Λ : existence (4)

Estimate for the difference: As for dyadic partitions we have

$$M_{ts}^{\hat{\pi}} = M_{ts}^{\pi_n} - (\delta B)_{r_{l+1}^n, r_l^n, r_{l-1}^n} = M_{st}^{\pi_n} - h_{r_{l+1}^n, r_l^n, r_{l-1}^n}.$$

and thus

$$\left| M_{ts}^{\hat{\pi}} - M_{ts}^{\pi_n} \right| \leq 2^\mu \|h\|_\mu \left(\frac{t-s}{k_n} \right)^\mu.$$

Iteration of the estimate: We repeat this operation and

- We end up with the trivial partition $\hat{\pi}_0 \equiv \{s, t\}$
- $M_{st}^{\hat{\pi}_0} = 0$
- We obtain

$$\left| M_{ts}^{\pi_n} \right| \leq 2^\mu \|h\|_\mu |t-s|^\mu \sum_{j=1}^{k_n} j^{-\mu} \leq 2^\mu \|h\|_\mu |t-s|^\mu \sum_{j=1}^{\infty} j^{-\mu} \equiv c_{\mu, h} |t-s|^\mu.$$

Operator Λ : existence (5)

More general sequences, conclusion: By compactness arguments

- One can find a subsequence $\{\pi_m; m \geq 1\}$ of $\{\pi_n; n \geq 1\}$
- It satisfies $\lim_{m \rightarrow \infty} M_{ts}^{\pi_m} = M_{ts}$
- M_{ts} , satisfies $M_{ts} \leq c_{\mu,h} |t - s|^\mu$

Uniqueness of the limit: One can show

\Leftrightarrow That the limit does not depend on the sequence of partitions.

Operator Λ : existence (6)

Algebraic property:

We wish to show that $\delta M = h$.

Family of partitions: Consider

- $0 \leq s < u < t \leq T$
- π_{su}^n sequence of partitions of $[s, u]$ such that $\lim_{n \rightarrow 0} |\pi_{su}^n| = 0$
- π_{ut}^n sequence of partitions of $[u, t]$ such that $\lim_{n \rightarrow 0} |\pi_{ut}^n| = 0$
- $\pi_{st}^n = \pi_{su}^n \cup \pi_{ut}^n$

Limits along the partitions: One can construct $\pi_{ut}^n, \pi_{su}^n, \pi_{st}^n$ such that

$$\lim_{m \rightarrow \infty} M_{ut}^{\pi_{ut}^n} = M_{ut}, \quad \lim_{m \rightarrow \infty} M_{su}^{\pi_{su}^n} = M_{su}, \quad \lim_{m \rightarrow \infty} M_{st}^{\pi_{st}^n} = M_{st}.$$

Operator Λ : existence (7)

Notation: We call

- k_{st}^n the number of points of the partition π_{st}^n
- k_{su}^n the number of points of the partition π_{su}^n
- k_{ut}^n the number of points of the partition π_{ut}^n

Applying δ : We have

$$\begin{aligned}\delta M_{sut}^{\pi_{st}^n} &= M_{st}^{\pi_{st}^n} - M_{su}^{\pi_{su}^m} - M_{ut}^{\pi_{ut}^m} \\ &= \delta B_{sut} - \left(\sum_{l=0}^{k_{st}^n + k_{su}^n - 1} B_{r_l^n r_{l+1}^n} - \sum_{l=0}^{k_{su}^n - 1} B_{r_l^n r_{l+1}^n} - \sum_{l=k_{su}^n}^{k_{su}^m + k_{ut}^m - 1} B_{r_l^m r_{l+1}^m} \right) \\ &= \delta B_{sut} = h_{sut}.\end{aligned}$$

Taking the limit $n \rightarrow \infty$ in the latter relation, we get $\delta M_{sut} = h_{sut}$

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration
- 3 Increments
- 4 Algebraic Young integration**
- 5 Differential equations

Expression for smooth functions

Riemann integral: Let $f, g \in \mathcal{C}_1^1$

$\hookrightarrow \mathcal{J}_{st}(f dg)$ defined in Riemann sense and

$$\begin{aligned}\mathcal{J}_{st}(f dg) &\equiv \int_s^t f_u dg_u = f_s \delta g_{st} + \int_s^t [f_u - f_s] dg_u \\ &= f_s \delta g_{st} + \int_s^t \delta f_{su} dg_u = f_s \delta g_{st} + \mathcal{J}_{st}(\delta f dg).\end{aligned}$$

Analysis of $\mathcal{J}(\delta f dg) \in \mathcal{C}_2$: for $s, u, t \in [0, T]$ we have

$$h_{sut} \equiv [\delta(\mathcal{J}(df dg))]_{sut} = \delta f_{su} \delta g_{ut}.$$

Therefore, $f \in \mathcal{C}_1^\kappa, g \in \mathcal{C}_1^\gamma \Rightarrow h \in \mathcal{ZC}_3^{\gamma+\kappa}$

Expression for smooth functions (2)

We have seen:

If $\kappa + \gamma > 1$ (smooth case: $\kappa = \gamma = 1$), then $h \in \text{Dom}(\Lambda)$

Thus (explain convention on products),

$$\mathcal{J}(\delta f dg) = \Lambda(h) = \Lambda(\delta f \delta g),$$

and we get:

$$\mathcal{J}_{st}(f dg) = f_s \delta g_{st} + \Lambda_{st}(\delta f \delta g). \quad (9)$$

Generalization: RHS in (9) makes sense whenever $\kappa + \gamma > 1$
 \hookrightarrow natural extension of the notion of integral

Theorem 12.

Let $f \in \mathcal{C}_1^\kappa, g \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$. Define

$$\mathcal{J}_{st}(f dg) = f_s \delta g_{st} + \Lambda_{st}(\delta f \delta g). \quad (10)$$

Then:

- 1 If f, g are smooth functions
 \hookrightarrow Then $\mathcal{J}_{st}(f dg) =$ Riemann integral
- 2 Generalized integral $\mathcal{J}(f dg)$ satisfies:

$$|\mathcal{J}_{st}(f dg)| \leq \|f\|_\infty \|g\|_\gamma |t - s|^\gamma + c_{\gamma, \kappa} \|f\|_\kappa \|g\|_\gamma |t - s|^{\gamma + \kappa}.$$

- 3 $\mathcal{J}_{st}(f dg)$ coincides with usual Young integral:

$$\mathcal{J}_{st}(f dg) = \lim_{|\pi_{st}| \rightarrow 0} \sum_{i=0}^{n-1} f_{t_i} \delta g_{t_i t_{i+1}}.$$

Proof

Claim 1: Already obtained at (9)

Claim 2: Recall that

$$\mathcal{J}_{st}(f dg) = f_s \delta g_{st} + \Lambda_{st}(\delta f \delta g).$$

Hence, setting $h = \delta f \delta g$:

$$\begin{aligned} |f_s \delta g_{st}| &\leq \|f\|_\infty \|g\|_\gamma |t - s|^\gamma \\ |\Lambda_{st}(\delta f \delta g)| &\leq c_{\gamma, \kappa} \|h\|_{\gamma + \kappa} |t - s|^{\gamma + \kappa} \leq c_{\gamma, \kappa} \|f\|_\kappa \|g\|_\gamma |t - s|^{\gamma + \kappa} \end{aligned}$$

Proof (2)

Claim 3:

Recall: that, if $\delta\ell \in \mathcal{C}_3^\mu$ with $\mu > 1$,

$$k = (\text{Id} - \Lambda\delta)\ell \quad \Rightarrow \quad k_{st} = \lim_{|\pi_{st}| \rightarrow 0} \sum_{i=0}^n \ell_{t_i t_{i+1}}$$

Application: take $\ell = f\delta g$, namely $\ell_{st} = f_s \delta g_{st} \Rightarrow \delta\ell = -\delta f \delta g$

Conclusion: we have

$$\begin{aligned} f_s \delta g_{st} + \Lambda_{st}(\delta f \delta g) &= f_s \delta g_{st} - \Lambda_{st}(\delta(f \delta g)) = [\text{Id} - \Lambda\delta](f \delta g) \\ &= \lim_{|\pi_{st}| \rightarrow 0} \sum_{i=0}^n f_{t_i} \delta g_{t_i t_{i+1}} \end{aligned}$$

Outline

- 1 Some basic properties of fBm
- 2 Simple Young integration
- 3 Increments
- 4 Algebraic Young integration
- 5 Differential equations**

Pathwise strategy (repeated)

Aim: Let x be a function in \mathcal{C}_1^γ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s) dx_s \quad (11)$$

Steps:

- Define an integral $\int z_s dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (11) through fixed point argument in \mathcal{C}_1^κ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.

Existence-uniqueness result

Theorem 13.

Consider

- Noise: $x \in \mathcal{C}_1^\gamma \equiv \mathcal{C}_1^\gamma([0, T])$, with $\gamma > 1/2$
- Coefficient: $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ a C_b^2 function
- Equation: $\delta y = \mathcal{J}(\sigma(y) dx)$

Then:

- 1 Our equation admits a unique solution y in \mathcal{C}_1^κ for any $1/2 < \kappa < \gamma$.
- 2 Application $(a, x) \mapsto y$ is continuous from $\mathbb{R} \times \mathcal{C}_1^\gamma$ to \mathcal{C}_1^κ .

Fixed point: strategy

A map on a small interval:

Consider an interval $[0, \tau]$, with τ to be determined later

Consider κ such that $1/2 < \kappa < \gamma < 1$

In this interval, consider $\Gamma : \mathcal{C}_1^\kappa([0, \tau]) \rightarrow \mathcal{C}_1^\kappa([0, \tau])$ defined by:
 $\Gamma(z) = \hat{z}$, with $\hat{z}_0 = a$, and for $s, t \in [0, \tau]$:

$$\delta \hat{z}_{st} = \int_s^t \sigma(z_r) dx_r = \mathcal{J}_{st}(\sigma(z) dx)$$

Aim: See that for a small enough τ , the map Γ is a contraction
 \hookrightarrow our equation admits a unique solution in $\mathcal{C}_1^\kappa([0, \tau])$

Contraction argument in $[0, \tau]$

Definition of 2 processes:

Let $z^1, z^2 \in \mathcal{C}_1^\kappa([0, \tau])$. Define $\hat{z}^i = \Gamma(z^i)$. Then

$$\delta(\hat{z}^1 - \hat{z}^2)_{st} = \int_s^t [\sigma(z_r^1) - \sigma(z_r^2)] dx_s = \mathcal{J}_{st}([\sigma(z^1) - \sigma(z^2)] dx)$$

Evaluation of the difference:

$$\begin{aligned} \left| \mathcal{J}_{st}([\sigma(z^1) - \sigma(z^2)] dx) \right| &\leq \|\sigma(z^2) - \sigma(z^1)\|_\infty \|x\|_\gamma |t - s|^\gamma \\ &\quad + c_{\gamma, \kappa} \|\sigma(z^2) - \sigma(z^1)\|_\kappa \|x\|_\gamma |t - s|^{\gamma + \kappa} \end{aligned}$$

Important step: Control of

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa$$

Control of $\|\sigma(z^2) - \sigma(z^1)\|_\kappa$

Lemma 14.

Let $\sigma \in C_b^2$. We have

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa \leq c_{\sigma, \tau} \left(1 + \|z^1\|_\kappa + \|z^2\|_\kappa\right) \|z^2 - z^1\|_\kappa$$

Problem: Application $\sigma : C_1^\kappa([0, \tau]) \rightarrow C_1^\kappa([0, \tau])$
is only locally Lipschitz

Solution: Decomposition of the fixed point argument:

- 1 If τ small enough and M large enough:
existence of an invariant ball $B(0, M)$ by map Γ in $C_{1,a}^\kappa([0, \tau])$
- 2 Within the invariant ball, usual contraction argument

Invariant ball

Lemma 15.

Let $\tau < c^{-1/\gamma}$, where $c = c(\gamma, \kappa, \|\sigma\|_\infty, \|\sigma'\|_\infty, \|x\|_\gamma)$.

Let $M \geq [c\tau^{\gamma-\kappa}]/[1 - c\tau^\gamma]$.

Then ball $B(0, M)$ in $C_{1,a}^\kappa([0, \tau])$ is invariant by Γ .

Proof: Start from our bound:

$$\begin{aligned} |\mathcal{J}_{st}(\sigma(z)dx)| &\leq \|\sigma(z)\|_\infty \|x\|_\gamma |t - s|^\gamma + c_{\gamma,\kappa} \|\sigma(z)\|_\kappa \|x\|_\gamma |t - s|^{\gamma+\kappa} \\ &\leq \|\sigma\|_\infty \|x\|_\gamma |t - s|^\kappa \tau^{\gamma-\kappa} + c_{\gamma,\kappa} \|\sigma'\|_\infty \|z\|_\kappa \|x\|_\gamma |t - s|^\kappa \tau^\gamma \\ &\leq c_{\gamma,\kappa,\sigma} \|x\|_\gamma \left[\tau^{\gamma-\kappa} + \|z\|_\kappa \tau^\gamma \right] |t - s|^\kappa \\ &\leq c \left[\tau^{\gamma-\kappa} + \|z\|_\kappa \tau^\gamma \right] |t - s|^\kappa \end{aligned}$$

Invariant ball: proof

Inequality for M : We have seen

$$\|\Gamma(z)\|_{\kappa} \leq c \left[\tau^{\gamma-\kappa} + \|z\|_{\kappa} \tau^{\gamma} \right].$$

Hence, if M satisfies:

$$c \left[\tau^{\gamma-\kappa} + M \tau^{\gamma} \right] \leq M, \quad (12)$$

ball $B(0, M)$ invariant by Γ .

Remark:

We have used $\gamma > \kappa$ in order to gain a contraction factor $\tau^{\gamma-\kappa}$

Invariant ball: proof (2)

Solving (12): Write

$$(12) \iff c \left[\tau^{\gamma-\kappa} + M\tau^\gamma \right] \leq M \iff M(1 - c\tau^\gamma) \geq c\tau^{\gamma-\kappa}$$

First condition on τ : $c\tau^\gamma \leq \frac{1}{2}$. Then a sufficient condition for (12) is

$$M \geq 2c\tau^{\gamma-\kappa}$$

Second condition on τ : We take $M = 1$ and $\tau \leq \left(\frac{1}{2c}\right)^{\frac{1}{\gamma-\kappa}}$

Conclusion: Relation (12) satisfied and $B(0, 1)$ invariant if

$$\tau \leq \left(\frac{1}{2c}\right)^{\frac{1}{\gamma}} \wedge \left(\frac{1}{2c}\right)^{\frac{1}{\gamma-\kappa}} \equiv \tau_1(\gamma, \kappa, \sigma, \|x\|_\gamma)$$

Contraction argument in $[0, \tau_1]$

Recall: Setting $K_{st} \equiv \mathcal{J}_{st}([\sigma(z^1) - \sigma(z^2)] dx)$, we have seen

$$|K_{st}| \leq \|\sigma(z^2) - \sigma(z^1)\|_\infty \|x\|_\gamma |t - s|^\gamma \\ + c_{\gamma, \kappa} \|\sigma(z^2) - \sigma(z^1)\|_\kappa \|x\|_\gamma |t - s|^{\gamma + \kappa}$$

Bounds on Hölder norms:

On $[0, \tau_2]$ with $\tau_2 \leq \tau_1$ we have (cf Lemma 14)

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa \leq 3c_{\sigma, \tau_2} \|z^2 - z^1\|_\kappa$$

and

$$\begin{aligned} \|\sigma(z^2) - \sigma(z^1)\|_\infty &\leq c_\sigma \|z^2 - z^1\|_\infty \\ &\leq c_\sigma \tau_2^\kappa \|z^2 - z^1\|_\kappa \\ &= c_{\sigma, \tau_2} \|z^2 - z^1\|_\kappa \end{aligned}$$

Contraction argument in $[0, \tau_1]$ (2)

Bound on K : Owing to previous computations we get

$$|K_{st}| \leq c_{\sigma, \tau_2} \|x\|_{\gamma} \tau_2^{\kappa} \|z^2 - z^1\|_{\gamma} |t - s|^{\gamma}$$

Recall: Let $z^1, z^2 \in \mathcal{C}_1^{\kappa}([0, \tau_2])$. Define $\hat{z}^i = \Gamma(z^i)$. Then

$$\delta(\hat{z}^1 - \hat{z}^2)_{st} = K_{st}$$

Contraction: We have obtained

$$\|\Gamma(z^2) - \Gamma(z^1)\|_{\gamma} \leq c_{\sigma, \tau_2} \|x\|_{\gamma} \tau_2^{\kappa} \|z^2 - z^1\|_{\gamma}$$

Considering $\tau_2 \leq \inf\{\tau_1, (2c_{\sigma, \tau_2} \|x\|_{\gamma})^{-1/\kappa}\}$ this yields

$$\|\Gamma(z^2) - \Gamma(z^1)\|_{\gamma} \leq \frac{1}{2} \|z^2 - z^1\|_{\gamma}$$

Contraction argument in $[0, \tau_1]$ (3)

Existence-uniqueness on a small interval:

Thanks to Banach-Steinhaus, for

$$\tau_2 \leq \inf \left\{ \tau_1, \frac{1}{(2c_{\sigma, \tau_2} \|x\|_{\gamma})^{1/\kappa}} \right\},$$

we get unique solution of (11) in $\mathcal{C}_{1,a}^{\kappa}([0, \tau_2])$

From $[0, \tau]$ to $[\tau, 2\tau]$

New map Γ : In $[\tau, 2\tau]$, consider the map

$$\Gamma : \mathcal{C}_1^\kappa([\tau, 2\tau]) \rightarrow \mathcal{C}_1^\kappa([\tau, 2\tau])$$

defined by: $\Gamma(z) = \hat{z}$, with $\hat{z}_\tau = a_\tau$, where

- $a_\tau \equiv$ final value of the solution in $[0, \tau]$
- For $s, t \in [\tau, 2\tau]$, $\delta \hat{z}_{st} = \mathcal{J}_{st}(z dx)$

New fixed point argument: the same fixed point arguments yield a unique solution y of $y_t = a_\tau + \int_\tau^t f(y_s) dx_s$ in $\mathcal{C}_1^\kappa([\tau, 2\tau])$.

Remark:

In order to use the very same arguments, need a bound on $\sigma, \sigma', \sigma''$

Continuity with respect to initial condition (1)

Notation: We set

- y^a solution of equation (11) with initial condition a
- a_1, a_2 two initial conditions
- $z = y^{a_2} - y^{a_1}$

Equation for z :

$$\delta z_{st} = [\sigma(y_u^{a_1}) - \sigma(y_u^{a_2})] \delta x_{st} + \Lambda(\delta[\sigma(y_u^{a_1}) - \sigma(y_u^{a_2})]) \delta x$$

Continuity with respect to initial condition (2)

Notation: Set, for $\tau_1 > 0$,

- $\|w\|_\gamma = \|w\|_{\gamma, [0, \tau_1]}$ for a path w
- $Z_s = \sup_{r \leq s} |z_s|$
- $c_1 = c_\sigma \|x\|_\gamma$
- $c_2 = c_{\kappa, \gamma, \sigma} (1 + \|y^{a^1}\|_\kappa + \|y^{a^2}\|_\kappa) \|x\|_\gamma$

Bound for z : We get

$$\|\delta z_{st}\| \leq c_1 Z_s |t - s|^\gamma + c_2 \|z\|_\kappa |t - s|^{\gamma + \kappa}$$

Bound for Z : We trivially have

$$Z_s \leq |a^1 - a^2| + \|z\|_\kappa \tau_1^\kappa$$

Continuity with respect to initial condition (3)

Bound for the Hölder norm of z : We have

$$\begin{aligned}\|z\|_{\kappa} &\leq c_1 \tau_1^{\gamma-\kappa} (|a^1 - a^2| + \|z\|_{\kappa} \tau_1^{\kappa}) + c_2 \|z\|_{\kappa} \tau_1^{\gamma} \\ &\leq c_1 \tau_1^{\gamma-\kappa} |a^1 - a^2| + (c_1 + c_2) \tau_1^{\gamma} \|z\|_{\kappa}\end{aligned}$$

Choosing τ_1 : such that $\tau_1 \leq 1$ and

$$\tau_1 = \left(\frac{c_3}{1 + \|x\|_{\gamma}} \right)^{\frac{1}{\gamma}} \implies (c_1 + c_2) \tau_1^{\gamma} = \frac{1}{2}$$

Conclusion on a small interval: On $[0, \tau_1]$ we have

$$\begin{aligned}\|z\|_{\kappa; [0, \tau_1]} &\leq 2c_1 \tau_1^{\gamma-\kappa} |a^1 - a^2| \\ |z_{\tau_1}| &\leq |z_0| + \tau_1^{\gamma} \|z\|_{\gamma; [0, \tau_1]} \leq (1 + c_4 \tau_1^{\gamma}) |a^1 - a^2|\end{aligned}$$

Continuity with respect to initial condition (4)

Iteration of the estimate: For $j \geq 0$ and setting $d_x = 1 + c_4 \tau_1^\gamma$ we get

$$\|z\|_{\kappa; [j\tau_1, (j+1)\tau_1]} \leq d_x^j |a^1 - a^2|$$

Patching small interval estimates: Consider

$$j\tau_1 \leq s < (j+1)\tau_1 < k\tau_1 \leq t < (k+1)\tau_1$$

Then

$$|\delta z_{st}| \leq |\delta z_{s, (j+1)\tau_1}| + \sum_{l=j+1}^{k-1} |\delta z_{l\tau_1, (l+1)\tau_1}| + |\delta z_{k\tau_1, t}|$$

Continuity with respect to initial condition (5)

Patching small interval estimates, ctd:

We get (recall $d_x - 1 = c_4 \tau_1^\gamma$)

$$\begin{aligned} \frac{|\delta z_{st}|}{|a^1 - a^2|} &\leq d_x^j |(j+1)\tau_1 - s|^\gamma + \sum_{l=j+1}^{k-1} d_x^l \tau_1^\gamma + d_x^k |t - k\tau_1|^\gamma \\ &\leq d_x^j |(j+1)\tau_1 - s|^\gamma + d_x^{j+1} \frac{d_x^{k-j-1} - 1}{d_x - 1} \tau_1^\gamma + d_x^k |t - k\tau_1|^\gamma \\ &\leq c_5 d_x^k (|(j+1)\tau_1 - s|^\gamma + \tau_1^\gamma + |t - k\tau_1|^\gamma) \\ &\leq c_6 d_x^k |t - s|^\gamma \end{aligned}$$

Continuity with respect to initial condition (6)

Bound on k : In previous computations,

$$k\tau_1 \leq T \quad \Longrightarrow \quad k \leq \frac{T}{\tau_1}$$

Conclusion for Hölder's norm: We have obtained

$$\begin{aligned} \|z\|_{\gamma; [0, T]} &\leq c_6 d_x^{T/\tau_1} |a^1 - a^2| = c_6 \exp\left(\frac{T}{\tau_1} \ln(d_x)\right) |a^1 - a^2| \\ &\leq c_6 \exp\left(c_7 (1 + \|x\|_\gamma)^{1/\gamma}\right) |a^1 - a^2| \end{aligned}$$

Continuity proved!

Control of $\|\sigma(z^2) - \sigma(z^1)\|_{\kappa}$ (repeated)

Lemma 16.

Let $\sigma \in C_b^2$. We have

$$\|\sigma(z^2) - \sigma(z^1)\|_{\kappa} \leq c_{\sigma, \tau} \left(1 + \|z^1\|_{\kappa} + \|z^2\|_{\kappa}\right) \|z^2 - z^1\|_{\kappa}$$

Proof: For $\lambda, \mu \in [0, 1]$, define the path

$$a(\lambda, \mu) = z_s^1 + \lambda(z_t^1 - z_s^1) + \mu(z_s^2 - z_s^1) + \lambda\mu(z_t^2 - z_s^2 - z_t^1 + z_s^1)$$

Then

$$a(0, 0) = z_s^1, \quad a(0, 1) = z_s^2, \quad a(1, 0) = z_t^1, \quad a(1, 1) = z_t^2$$

Proof

Let $G(\lambda, \mu) \equiv \sigma(a(\lambda, \mu))$, and

$$\Delta_{st}^{12} \equiv [\sigma(z_t^2) - \sigma(z_t^1)] - [\sigma(z_s^2) - \sigma(z_s^1)]$$

We have:

$$\Delta_{st}^{12} = G(1, 1) - G(1, 0) - G(0, 1) + G(0, 0) = \int_0^1 \int_0^1 \partial_{\lambda, \mu}^2 G \, d\lambda d\mu$$

Set $\hat{z} \equiv z^2 - z^1$ and compute:

$$\partial_{\lambda, \mu}^2 G = \partial_{\lambda, \mu}^2 a \sigma'(a) + \partial_\lambda a \partial_\mu a \sigma''(a)$$

$$\partial_\lambda a = \mu \delta z_{st}^2 + [1 - \mu] \delta z_{st}^1$$

$$\partial_\mu a = \lambda \hat{z}_t + [1 - \lambda] \hat{z}_s$$

$$\partial_{\lambda, \mu}^2 a = \delta \hat{z}_{st}$$

Proof (2)

Thus:

$$|\partial_\lambda \mathbf{a}| = \left| \mu \delta \mathbf{z}_{st}^2 + [1 - \mu] \delta \mathbf{z}_{st}^1 \right| \leq \left[\|\mathbf{z}^1\|_\kappa + \|\mathbf{z}^2\|_\kappa \right] |t - s|^\kappa$$

$$|\partial_\mu \mathbf{a}| = \left| \lambda \hat{\mathbf{z}}_t + [1 - \lambda] \hat{\mathbf{z}}_s \right| \leq \|\mathbf{z}^1 - \mathbf{z}^2\|_\kappa \tau^\kappa$$

$$|\partial_{\lambda, \mu}^2 \mathbf{a}| = |\delta \hat{\mathbf{z}}_{st}| \leq \|\mathbf{z}^1 - \mathbf{z}^2\|_\kappa |t - s|^\kappa,$$

and

$$\begin{aligned} \partial_{\lambda, \mu}^2 G &= \left| \partial_{\lambda, \mu}^2 \mathbf{a} \sigma'(\mathbf{a}) + \partial_\lambda \mathbf{a} \partial_\mu \mathbf{a} \sigma''(\mathbf{a}) \right| \\ &\leq \left| \partial_{\lambda, \mu}^2 \mathbf{a} \right| \|\sigma'\|_\infty + |\partial_\lambda \mathbf{a}| |\partial_\mu \mathbf{a}| \|\sigma''\|_\infty \end{aligned}$$

The result is now easily deduced.