Rough paths methods 3: Second order structures

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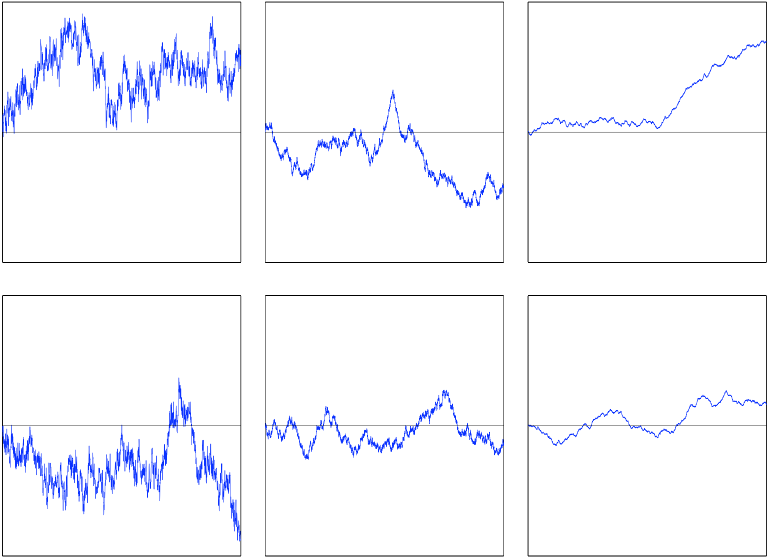
Outline

1. Heuristics
2. Controlled processes
3. Differential equations
4. Additional remarks
   - Other rough paths formalisms
   - Higher order structures

Outline

Examples of fBm paths



$H = 0.3$ $H = 0.5$ $H = 0.7$

General strategy

Aim: Define and solve an equation of the type:
$y_t = a + \int_0^t \sigma(y_s)\, dB_s$, where B is fBm.

Properties of fBm:
Generally speaking, take advantage of two aspects of fBm:

- Gaussianity
- Regularity

Remark: For $1/3 < H < 1/2$, Young integral isn't suficient

Levy area: We shall see that the following exists:
$\mathbf{B}^{\mathbf{2},ij}_{st} = \int_s^t dB^i_u \int_s^u dB^j_v \in \mathcal{C}_2^{2\gamma}$ for $\gamma < H$

Strategy: Given B and $\mathbf{B}^{\mathbf{2}}$ solve the equation in a pathwise manner

Pathwise strategy

Aim: For $x \in \mathcal{C}_1^\gamma$ con $1/3 < \gamma < 1/2$, define and solve an equation of the type:

$$y_t = a + \int_0^t \sigma(y_u)\, dx_u \tag{1}$$

Main steps:

- Define an integral $\int z_s\, dx_s$ for z: function whose increments are controlled by those of x
- Solve (1) by fixed point arguments in the class of controlled processes

Remark:

Like in the previous chapters, we treat a real case and $b \equiv 0$ for notational sake.

Caution: d-dimensional case really different here, because of $\mathbf{x}^2$

Heuristics (1)

Hypothesis:
Solution y_t exists in a space $\mathcal{C}_1^\gamma([0,T])$

A priori decomposition for y:

$$\begin{aligned} \delta y_{st} &\equiv y_t - y_s = \int_s^t \sigma(y_v) dx_v \\ &= \sigma(y_s)\,\delta x_{st} + \int_s^t [\sigma(y_v) - \sigma(y_s)] dx_v \\ &= \zeta_s\,\delta x_{st} + r_{st} \end{aligned}$$

Expected coefficients regularity:
$\zeta = \sigma(y)$: bounded, γ-Hölder, r: 2γ-Hölder

Heuristics (2)

Start from controlled structure: Let z such that

$$\delta z_{st} = \zeta_s\, \delta x_{st} + r_{st}, \quad \text{with} \quad \zeta \in \mathcal{C}^{\gamma},\ r \in \mathcal{C}^{2\gamma} \tag{2}$$

Formally:

$$\begin{aligned}
\int_s^t z_v\, dx_v &= z_s\, \delta x_{st} + \int_s^t \delta z_{sv}\, dx_v \\
&= z_s\, \delta x_{st} + \zeta_s \int_s^t \delta x_{sv}\, dx_v + \int_s^t r_{sv}\, dx_v \\
&= z_s\, \delta x_{st} + \zeta_s\, \mathbf{x}^{\mathbf{2}}_{st} + \int_s^t r_{sv}\, dx_v
\end{aligned}$$

Heuristics (3)

Formally, we have seen: z satisfies

$$\int_s^t z_v \, dx_v = z_s \, \delta x_{st} + \zeta_s \mathbf{x}^{\mathbf{2}}_{st} + \int_s^t r_{sv} \, dx_v$$

Integral definition:

- $z_s \, \delta x_{st}$ trivially defined
- $\zeta_s \mathbf{x}^{\mathbf{2}}_{st}$ well defined, if Levy area $\mathbf{x}^{\mathbf{2}}$ provided
- $\int_s^t r_{sv} dB_v$ defined through operator Λ if $r \in \mathcal{C}_2^{2\gamma}$, $x \in \mathcal{C}_1^{\gamma}$ and $3\gamma > 1$

Remark:

- We shall define $\int_s^t z_v \, dx_v$ more rigorously
- Equation (1) solved within class of proc. with decomposition (2)

Outline

1 Heuristics

2 Controlled processes

3 Differential equations

4 Additional remarks
- Other rough paths formalisms
- Higher order structures

Controlled processes

Definition 1.

Let

- $1/3 < \kappa \leq \gamma$
- $z \in \mathcal{C}_1^\kappa$

We say that z is a process controlled by x, if $z_0 = a \in \mathbb{R}$, and

$$\delta z = \zeta \delta x + r, \quad \text{i.e.} \quad \delta z_{st} = \zeta_s \, \delta x_{st} + r_{st}, \quad s, t \in [0, T], \tag{3}$$

with

- $\zeta \in \mathcal{C}_1^\kappa$
- r is a remainder such that $r \in \mathcal{C}_2^{2\kappa}$

Space of controlled processes

Definition 2.

Space of controlled processes:

- Denoted by $\mathcal{Q}_{\kappa,a}$
- $z \in \mathcal{Q}_{\kappa,a}$ should be considered as a couple (z, ζ)

Natural semi-norm on $\mathcal{Q}_{\kappa,a}$:

$$\mathcal{N}[z; \mathcal{Q}_{\kappa,a}] = \mathcal{N}[z; \mathcal{C}_1^{\kappa}] + \mathcal{N}[\zeta; \mathcal{C}_1^{b}] + \mathcal{N}[\zeta; \mathcal{C}_1^{\kappa}] + \mathcal{N}[r; \mathcal{C}_2^{2\kappa}]$$

with

- $\mathcal{N}[g; \mathcal{C}_1^{\kappa}] = \|g\|_{\kappa}$
- $\mathcal{N}[\zeta; \mathcal{C}_1^{b}(V)] = \sup_{0 \leq s \leq T} |\zeta_s|_V$

Operations on controlled processes

In order to solve equations, two preliminary steps:

1. Study of transformation $z \mapsto \varphi(z)$ for
    - Controlled process z
    - Smooth function φ
2. Integrate controlled processes with respect to x

Composition of controlled processes

Proposition 3.

Consider $z \in \mathcal{Q}_{\kappa,a}$, $\varphi \in C_b^2$. Define

$$\hat{z} = \varphi(z), \quad \hat{a} = \varphi(a).$$

Then $\hat{z} \in \mathcal{Q}_{\kappa,\hat{a}}$, and

$$\delta\hat{z} = \hat{\zeta}\delta x + \hat{r},$$

with

$$\hat{\zeta} = \nabla\varphi(z)\zeta \quad \text{and} \quad \hat{r} = \nabla\varphi(z)r + [\delta(\varphi(z)) - \nabla\varphi(z)\delta z].$$

Furthermore, $\mathcal{N}[\hat{z}; \mathcal{Q}_{\kappa,\hat{a}}] \leq c_{\varphi,T}\left(1 + \mathcal{N}^2[z; \mathcal{Q}_{\kappa,a}]\right)$.

Proof

Algebraic part: Just write

$$
\begin{aligned}
\delta\hat{z}_{st} &= \varphi(z_t) - \varphi(z_s) \\
&= \nabla\varphi(z_s)\delta z_{st} + \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)\delta z_{st} \\
&= \nabla\varphi(z_s)\zeta_s\delta x_{st} + \nabla\varphi(z_s) r_{st} + \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)\delta z_{st} \\
&= \hat{\zeta}_s\delta x_{st} + \hat{r}_{st}
\end{aligned}
$$

Proof (2)

Bound for $\mathcal{N}[\hat{z};\, \mathcal{Q}_{\kappa,\hat{a}}(\mathbb{R}^n)]$, strategy: get bound on

- $\mathcal{N}[\hat{z}; \mathcal{C}_1^{\kappa}(\mathbb{R}^n)]$
- $\mathcal{N}[\hat{\zeta}; \mathcal{C}_1^{\kappa}\mathcal{L}^{d,n}]$
- $\mathcal{N}[\hat{\zeta}; \mathcal{C}_1^{b}\mathcal{L}^{d,n}]$
- $\mathcal{N}[\hat{r}; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)]$

Decomposition for $\hat{r}$: We have

$$\hat{r} = \hat{r}^1 + \hat{r}^2$$

with

$$\hat{r}^1_{st} = \nabla\varphi(z_s) r_{st} \quad \text{and} \quad \hat{r}^2_{st} = \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)(\delta z)_{st}. \quad (4)$$

Proof (3)

Bound for $\hat{r}^1$: $\nabla\varphi$ is a bounded $\mathcal{L}^{k,n}$-valued function. Therefore

$$\mathcal{N}[\hat{r}^1; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \leq \|\nabla\varphi\|_\infty \mathcal{N}[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]. \quad (5)$$

Bound for $\hat{r}^2$:

$$|\hat{r}^2_{st}| \leq \frac{1}{2}\|\nabla^2\varphi\|_\infty |(\delta z)_{st}|^2 \leq c_\varphi \mathcal{N}^2[z; \mathcal{C}_1^\kappa(\mathbb{R}^k)]|t-s|^{2\kappa},$$

which yields

$$\mathcal{N}[\hat{r}^2; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \leq c_\varphi \mathcal{N}^2[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)], \quad (6)$$

Bound for $\hat{r}$: Since $\hat{r} = \hat{r}^1 + \hat{r}^2$, we get from (5) and (6)

$$\mathcal{N}[\hat{r}; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \leq c_\varphi \left(1 + \mathcal{N}^2[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]\right)$$

Proof (4)

Other estimates: We still have to bound

- $\mathcal{N}[\hat{z}; \mathcal{C}_1^\kappa(\mathbb{R}^n)]$
- $\mathcal{N}[\hat{\zeta}; \mathcal{C}_1^\kappa \mathcal{L}^{d,n}]$
- $\mathcal{N}[\hat{\zeta}; \mathcal{C}_1^b \mathcal{L}^{d,n}]$

Done in the same way as for $\hat{r}$

Conclusion for the analytic part: We obtain

$$\mathcal{N}[\hat{z}; \mathcal{Q}_{\kappa,\hat{a}}] \leq c_{\varphi,T} \left(1 + \mathcal{N}^2[z; \mathcal{Q}_{\kappa,a}]\right)$$

Composition of controlled processes (ctd)

Remark: In previous proposition

- Quadratic bound instead of linear as in the Young case
- Due to Taylor expansions of order 2

Next step: Define $\mathcal{J}(z\,dx)$ for a controlled process z:

- Start with smooth x, z
- Try to recast $\mathcal{J}(z\,dx)$ with expressions making sense for a controlled process $z \in \mathcal{C}_1^{\kappa}$

Integration of smooth controlled processes

Hypothesis:

- x, ζ smooth functions, r smooth increment
- Smooth controlled process $z \in \mathcal{Q}_{1,a}$, namely $\delta z_{st} = \zeta_s \, \delta x_{st} + r_{st}$

Expression of the integral: $\mathcal{J}(z\,dx)$ defined as Riemann integral and

$$\int_s^t z_u dx_u = z_s[x_t - x_s] + \int_s^t [z_u - z_s] dx_u$$

Otherwise stated:

$$\mathcal{J}(z\,dx) = z\,\delta x + \mathcal{J}(\delta z\,dx).$$

Integration of smooth controlled processes (2)

Levy area shows up: if $\delta z_{st} = \zeta_s \, \delta x_{st} + r_{st}$,

$$\mathcal{J}(z\,dx) \;=\; z\,\delta x + \mathcal{J}(\zeta \delta x\,dx) + \mathcal{J}(r\,dx). \qquad (7)$$

Transformation of $\mathcal{J}(\zeta\delta x\,dx)$:

$$\mathcal{J}_{st}(\zeta\delta x\,dx) = \int_s^t \zeta_s \, [\delta x_{su} dx_u] = \zeta_s \mathbf{x}^{\mathbf{2}}_{st}$$

Plugging in (7) we get

$$\mathcal{J}(z\,dx) = z\,\delta x + \zeta\,\mathbf{x}^{\mathbf{2}} + \mathcal{J}(r\,dx)$$

Multidimensional case:

$$\int_s^t \zeta_s \, [\delta x_{su}\, dx_u] \longleftrightarrow \int_s^t \zeta_s^{ij} \, \left[\delta x^j_{su}\, dx^i_u\right] = \zeta_s^{ij}\, \mathbf{x}^{\mathbf{2},ji}_{st}$$

Levy area

Recall: $\mathcal{J}(z\,dx) = z\,\delta x + \zeta\,\mathbf{x^2} + \mathcal{J}(r\,dx)$
$\hookrightarrow$ For $\gamma < 1/2$, $\mathbf{x^2}$ enters as an additional data

Hypothesis 4.

Path x is γ-Hölder with $\gamma > 1/3$, and admits a Levy area, i.e

$$\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d}), \quad \text{formally defined as} \quad \mathbf{x^2} = "\mathcal{J}(dxdx)",$$

and satisfying:

$$\delta\mathbf{x^2} = \delta x \otimes \delta x, \quad \text{i.e.} \quad \delta\mathbf{x}^{\mathbf{2},ij}_{sut} = \delta x^i_{su}\,\delta x^j_{ut},$$

for any $s, u, t \in \mathcal{S}_{3,T}$ and $i, j \in \{1, \ldots, d\}$.

Levy area: particular cases

Levy area defined in following cases:

1. x is a regular path
   $\hookrightarrow$ Levy area defined in the Riemann sense
2. x is a fBm with $H > \frac{1}{4}$
   $\hookrightarrow$ Levy area defined in the Stratonovich sense

Integration of smooth controlled processes (3)

Analysis of $\mathcal{J}(r\,dx)$: we have seen

$$\mathcal{J}(r\,dx) = \mathcal{J}(z\,dx) - z\,\delta x - \zeta\,\mathbf{x}^{\mathbf{2}}$$

Apply δ on each side of the identity:

$$\begin{aligned}
&[\delta(\mathcal{J}(r\,dx))]_{sut} \\
&= \delta z_{su}\,\delta x_{ut} + \delta\zeta_{su}\,\mathbf{x}^{\mathbf{2}}_{ut} - \zeta_s\,\delta\mathbf{x}^{\mathbf{2}}_{sut} \\
&= \zeta_s\,\delta x_{su}\,\delta x_{ut} + r_{su}\,\delta x_{ut} + \delta\zeta_{su}\,\mathbf{x}^{\mathbf{2}}_{ut} - \zeta_s\,\delta x_{su}\,\delta x_{ut} \\
&= r_{su}\,\delta x_{ut} + \delta\zeta_{su}\,\mathbf{x}^{\mathbf{2}}_{ut}.
\end{aligned}$$

Integration of smooth controlled processes (4)

Recall: We have found

$$\delta(\mathcal{J}(r\,dx)) = r\,\delta x + \delta\zeta\,\mathbf{x^2}$$

Regularities: We have

- $r \in \mathcal{C}_2^{2\kappa}$
- $\delta x \in \mathcal{C}_2^{\gamma}$
- $\delta\zeta \in \mathcal{C}_2^{\kappa}$
- $\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}$

Since $\kappa + 2\gamma > 2\kappa + \gamma > 1$, Λ can be applied

Expression with Λ: We obtain

$$\delta(\mathcal{J}(r\,dx)) = r\,\delta x + \delta\zeta\,\mathbf{x^2} \quad \Longrightarrow \quad \mathcal{J}(r\,dx) = \Lambda(r\,\delta x + \delta\zeta\,\mathbf{x^2})$$

Integration of smooth controlled processes (5)

Conclusion: We have seen:

$$\begin{aligned}\mathcal{J}(z\,dx) &= z\,\delta x + \zeta\,\mathbf{x^2} + \mathcal{J}(r\,dx)\\ \mathcal{J}(r\,dx) &= \Lambda(r\,\delta x + \delta\zeta\,\mathbf{x^2})\end{aligned}$$

Thus, if m, x are smooth paths:

$$\mathcal{J}(z\,dx) = z\,\delta x + \zeta\,\mathbf{x^2} + \Lambda(r\,\delta x + \delta\zeta\,\mathbf{x^2})$$

Substantial gain: This expression can be extended to irregular paths!

Integration of controlled processes

Theorem 5.

Let

- $x \in \mathcal{C}_1^{\gamma}$, with $1/3 < \kappa < \gamma$
- x satisfies Hypothesis 4, with Levy area $\mathbf{x}^2$
- $z \in \mathcal{Q}_{\kappa,b}$, with decomposition $\delta z_{st} = \zeta_s \delta x_{st} + r_{st}$

Define ℓ by $z_0 = a \in \mathbb{R}$, and

$$\delta \ell \equiv \mathcal{J}(z\,dx) = z\,\delta x + \zeta \cdot \mathbf{x}^2 + \Lambda(r\,\delta x + \delta\zeta \cdot \mathbf{x}^2).$$

Then

1. ℓ is an element of $\mathcal{Q}_{\kappa,a}$
2. $\ell = \int z\,dx$ for smooth paths

Proof

Item 1: We have

- $\delta\ell = \zeta^{\ell}\delta x + r^{\ell}$
- $\zeta^{\ell} = z$
- $r^{\ell} = \zeta\,\mathbf{x}^{\mathbf{2}} + \Lambda(r\,\delta x + \delta\zeta\,\mathbf{x}^{\mathbf{2}})$

Item 2:

Proved in preliminary computations

Properties of the integral

Proposition 6.

Let ℓ be defined as in Theorem 5. Then on an interval $[0, \tau]$:

1. The semi-norm of ℓ in $\mathcal{Q}_{\kappa,a}$ satisfies

$$\mathcal{N}[\ell; \mathcal{Q}_{\kappa,a}] \leq c_x \left(|a| + \tau^{\gamma-\kappa} \mathcal{N}[z; \mathcal{Q}_{\kappa,a}] \right)$$

2. We have

$$\mathcal{J}_{st}(z\, dx) = \lim_{|\pi_{st}| \to 0} \sum_{i=0}^{n} \left[z_{t_i} \delta x_{t_i, t_{i+1}} + \zeta_{t_i} \cdot \mathbf{x}^{\mathbf{2}}_{t_i, t_{i+1}} \right]$$

Proof

Item 1: Elementary computations using decomposition

- $\delta\ell = \zeta^{\ell}\delta x + r^{\ell}$
- $\zeta^{\ell} = z$
- $r^{\ell} = \zeta\, \mathbf{x}^{2} + \Lambda(r\, \delta x + \delta\zeta\, \mathbf{x}^{2})$

Example of computation: Bound for $\zeta^{\ell} = z$. We have

$$|\delta z_{st}| \leq \|\zeta\|_{\infty}\|x\|_{\gamma}|t-s|^{\gamma} + \|r\|_{2\gamma}|t-s|^{2\gamma}$$

Hence

$$\|z\|_{\kappa} \leq \tau^{\gamma-\kappa}\left[\|\zeta\|_{\infty}\|x\|_{\gamma} + \tau^{\gamma}\|r\|_{2\gamma}\right] \leq c_{x}\tau^{\gamma-\kappa}\mathcal{N}[z; \mathcal{Q}_{\kappa,a}]$$

and

$$\|z\|_{\infty} \leq |z_0| + \tau^{\kappa}\|z\|_{\kappa} \leq c_{T}\left(|a| + \mathcal{N}[z; \mathcal{Q}_{\kappa,a}]\right)$$

Proof (2)

Recall: Let $g \in \mathcal{C}_2$, such that $\delta g \in \mathcal{C}_3^{\mu}$ with $\mu > 1$. Define

$$k = (\mathrm{Id} - \Lambda\delta) g$$

Then

$$k_{st} = \lim_{|\Pi_{st}| \to 0} \sum_{i=0}^{n} g_{t_i \, t_{i+1}},$$

as $|\Pi_{st}| \to 0$, where Π_{st} is a partition of $[s, t]$.

Proof (2)

Item 2: Let $g = z\delta x + \zeta \cdot \mathbf{x}^2$. Then

- $\delta g = -\left(r\,\delta x + \delta\zeta\,\mathbf{x}^2\right)$
- $\delta g \in \mathcal{C}_3^{3\kappa}$
- $\mathcal{J}(z\,dx) = (\mathrm{Id} - \Lambda\delta)g$

Therefore

$$\mathcal{J}_{st}(z\,dx) = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n} g_{t_i\,t_{i+1}},$$

which yields Item 2

Outline

Pathwise strategy

Hypothesis: x is a function of $\mathcal{C}_1^\gamma$ with $1/3 < \gamma \leq 1/2$.
It x admits a Levy area $\mathbf{x^2}$

Aim: We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s \tag{8}$$

Meaning of the equation: $y \in \mathcal{Q}_{a,\kappa}$, and

$$\delta y = \mathcal{J}(\sigma(y)\, dx)$$

Fixed point: strategy

A map on a small interval:

Consider an interval $[0, \tau]$, with τ to be determined later

Consider κ such that $1/2 < \kappa < \gamma < 1$

In this interval, consider $\Gamma : \mathcal{Q}_{a,\kappa}([0, \tau]) \to \mathcal{Q}_{a,\kappa}([0, \tau])$ defined by:
$\Gamma(z) = \hat{z}$, with $\hat{z}_0 = a$, and for $s, t \in [0, \tau]$:

$$\delta\hat{z}_{st} = \int_s^t \sigma(z_r) dx_r = \mathcal{J}_{st}(\sigma(z)\, dx)$$

Aim: See that for a small enough τ, the map Γ is a contraction
$\hookrightarrow$ our equation admits a unique solution in $\mathcal{C}_1^\kappa([0, \tau])$

Remark: Same kind of computations as in the Young case
$\hookrightarrow$ but requires more work (quadratic estimates, patching)!

Existence-uniqueness theorem

Theorem 7.

Let $x \in \mathcal{C}_1^\gamma$, with $1/3 < \kappa < \gamma$ and Levy area $\mathbf{x^2}$.
Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C_b^3 function. Then

1. Equation $\delta y = \mathcal{J}(\sigma(y)\, dx)$ admits a unique solution y in $\mathcal{Q}_{\kappa,a}$ for any $1/3 < \kappa < \gamma$.
2. Application $(a, x, \mathbf{x^2}) \mapsto y$ is continuous from $\mathbb{R} \times \mathcal{C}_1^\gamma \times \mathcal{C}_2^{2\gamma}$ to $\mathcal{Q}_{\kappa,a}$.

Proof

Bound on Γ: Set $\hat{z} = \Gamma(z)$ and $\hat{a} = \sigma(a)$.
Then according to Proposition 6,

$$\mathcal{N}[\hat{z};\, \mathcal{Q}_{\kappa,a}] \le c_x \left(|\hat{a}| + \tau^{\gamma-\kappa} \mathcal{N}[\sigma(z);\, \mathcal{Q}_{\kappa,\hat{a}}] \right).$$

Now thanks to Proposition 3,

$$\mathcal{N}[\hat{z};\, \mathcal{Q}_{\kappa,a}] \le c_x \left[|\hat{a}| + c_{\sigma,T} \tau^{\gamma-\kappa} \left(1 + \mathcal{N}^2[z;\, \mathcal{Q}_{\kappa,a}] \right) \right],$$

and thus

$$\mathcal{N}[\hat{z};\, \mathcal{Q}_{\kappa,a}] \le c_{\sigma,x} \left(1 + \tau^{\gamma-\kappa} \mathcal{N}^2[z;\, \mathcal{Q}_{\kappa,a}] \right) \qquad (9)$$

Proof (2)

Invariant set: For $\tau > 0$ set

$$\mathcal{A}_\tau = \left\{ u \in \mathbb{R}_+^* : \ c_{\sigma,x}(1 + \tau^{\gamma-\kappa} u^2) \leq u \right\}$$

Then

1. If τ small enough, $\mathcal{A}_\tau$ is non empty
2. In such case, consider $M \in \mathcal{A}_\tau$

Invariant ball: For τ_1 small enough and $M \in \mathcal{A}_{\tau_1}$, we have

$$B(0, M) \subset \mathcal{Q}_{\kappa,a} \quad \text{left invariant by} \quad \Gamma$$

Contraction within $B(0, M)$: Similar to Young case
$\hookrightarrow$ Gives existence-uniqueness on $[0, \tau]$ with $\tau = \tau_1 \wedge \tau_2$

Proof (3)

Patching small intervals:

On $[\tau, \tau_1]$, the key estimate is

$$\mathcal{N}[\hat{z};\, \mathcal{Q}_{\kappa,a}] \leq c_x \left[|\hat{a}| + c_{\sigma,T} \tau_1^{\gamma-\kappa} \left(1 + \mathcal{N}^2[z;\, \mathcal{Q}_{\kappa,a}] \right) \right],$$

where now

$$\hat{a} = \sigma(y_\tau) \quad \Longrightarrow \quad |\hat{a}| \leq \|\sigma\|_\infty$$

One can thus proceed as on $[0, \tau]$

Remark:

σ with linear growth out of scope of rough paths theory

Outline

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Lyons theory: Geometrical structures

Lie algebra: In general $(1, \mathbf{X^1}, \ldots, \mathbf{X^n}) \in \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^n$
$\hookrightarrow$ Lie algebra structure and associated Lie group: $G^n(\mathbb{R}^d)$
$\hookrightarrow$ Structures introduced by Chen in the '50s

Rough path: γ-Hölder function with values in $G^n(\mathbb{R}^d)$

Two important relations:
- $(1, \mathbf{X^1}, \ldots, \mathbf{X^n})$ determines all the iterated integrals if $n \geq \lfloor 1/\gamma \rfloor$
- Any element of $G^n(\mathbb{R}^d)$ can be realized as iterated integrals of a smooth function

Solving equations: Two possibilities
- Show that (y, x) is a single rough path
- Approximations, due to the second important relation above

Lyons theory vs. algebraic integration

Advantages of Lyons' approach:

- Elegant formalism (mixing geometry, analysis, probability)
- Approximation in $G^n(\mathbb{R}^d)$ yields powerful estimates:
  - Moments of solution to RDEs
  - Differential of RDEs

Advantages of algebraic integration:

- Simpler formalism
- Controlled process can be adapted easily to many situations:
  - Evolution, Volterra, Delay equations
  - Integration in the plane, SPDEs, Regularity structures
- Some results are hard to express without controlled processes:
  $\hookrightarrow$ Norris type lemma

Friz-Hairer's formalism

A short comparison with Friz-Hairer:

- Friz-Hairer's formalism also based on controlled processes
  $\hookrightarrow$ Reference to Gubinelli's derivative
- The use of δ, Λ is less explicit
  $\hookrightarrow$ In order to further simplify the theory
- Altogether, our presentation is very close to Friz-Hairer's book

Regularity structures

A brief summary of regularity structures:

Can be seen as a wide generalization of controlled rough paths

- Rough paths indexed by $\mathbb{R}^n$ (instead of $\mathbb{R}_+$)
- Richer rough paths structure indexed by trees (instead of $\mathbb{N}$)
- Product of distributions
- Additional group structure for renormalizations
- Evaluation of singularities

Typical example of equation related to regularity structures:

- Equation: $\partial_t Y_t(\xi) = \Delta Y_t(\xi) + (\partial_\xi Y_t(\xi))^2 + \dot{x}_t(\xi) - \infty$
- $(t, \xi) \in [0, 1] \times \mathbb{R}$
- $\dot{x} \equiv$ space-time white noise

Outline

Rough path assumptions

Regularity of X: $X \in \mathcal{C}^{\gamma}(\mathbb{R}^d)$ with $\gamma > 0$.

Iterated integrals: X allows to define

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \ldots, i_n) = \int_{s \le u_1 < \cdots < u_n \le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) \cdots dX_{u_n}(i_n),$$

for $0 \le s < t \le T$, $n \le \lfloor 1/\gamma \rfloor$ and $i_1, \ldots, i_n \in \{1, \ldots, d\}$.

Regularity of the iterated integrals: $\mathbf{X}^{\mathbf{n}} \in \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n})$, where

$$\mathcal{N}[g; \mathcal{C}_2^{\kappa}] \equiv \sup_{0 \le s < t \le T} \frac{|g_{st}|}{|t - s|^{\kappa}}$$

Main rough paths result

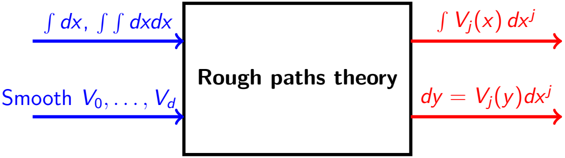
Theorem (loose formulation): Under the assumption of the previous slide, plus regularity assumptions on σ, one can

1. Obtain change of variables formula of Itô's type
2. Solve equations of the form $dY_t = \sigma(Y_t)dX_t$

Moreover, the application

$$\begin{aligned} F : \mathbb{R}^n \times \mathcal{C}_2^{\gamma}(\mathbb{R}^d) \times \cdots \times \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n}) &\longrightarrow \mathcal{C}^{\gamma}(\mathbb{R}^m) \\ (a, \mathbf{x}^1, \ldots, \mathbf{x}^n) &\mapsto Y \end{aligned}$$

is a continuous map



Meaning of the n^{th} iterated integral

Definition: The n^{th} order iterated integral associated to X is an element $\{\mathbf{X}^{\mathbf{n}}_{st}(i_1, \ldots, i_n);\, s \le t,\, 1 \le i_1, \ldots, i_n \le d\}$ satisfying:

(i) The regularity condition $\mathbf{X}^{\mathbf{n}} \in \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^n})$.

(ii) The multiplicative property:

$$\delta \mathbf{X}^{\mathbf{n}}_{sut}(i_1, \ldots, i_n) = \sum_{n_1=1}^{n-1} \mathbf{X}^{\mathbf{n_1}}_{su}(i_1, \ldots, i_{n_1}) \mathbf{X}^{\mathbf{n-n_1}}_{ut}(i_{n_1+1}, \ldots, i_n).$$

(iii) The geometric relation: $\mathbf{X}^{\mathbf{n}}_{st}(i_1, \ldots, i_n)\, \mathbf{X}^{\mathbf{m}}_{st}(j_1, \ldots, j_m)$ can be expressed in terms of higher order integrals

Remark: The notion of controlled process is also more complicated for higher order rough paths.