Rough paths methods 4: Application to fBm

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Outline

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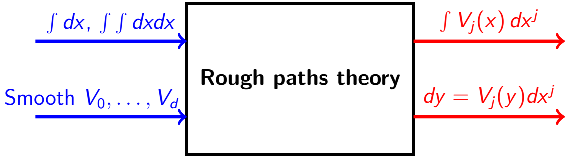
Objective

Summary: We have obtained the following picture



Remaining question:

How to define $\int\int dxdx$ when x is a fBm with $H \geq 1/2$?

Levy area of fBm

Proposition 1.

Let B be a d-dimensional fBm, with $H > 1/3$, and $1/3 < \gamma < H$. Almost surely, the paths of B:

1. Belong to $\mathcal{C}_1^\gamma$
2. Admit a Levy area $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}$ such that

$$\delta \mathbf{B^2} = \delta B \otimes \delta B, \quad \text{i.e.} \quad \mathbf{B}^{\mathbf{2},ij}_{sut} = \delta B^i_{su}\, \delta B^j_{ut}$$

Conclusion:

The abstract rough paths theory applies to fBm with $H > 1/3$

Proof of item 1: Already seen (Kolmogorov criterion)

Geometric and weakly geometric Levy area

Remark:

- The stack $\mathbf{B^2}$ as defined in Proposition 1 is called a weakly geometric second order rough path above X
  $\hookrightarrow$ allows a reasonable differential calculus
- When there exists a family B^ε such that
  - B^ε is smooth
  - $\mathbf{B}^{\mathbf{2},\varepsilon}$ is the iterated Riemann integral of B^ε
  - $\mathbf{B^2} = \lim_{\varepsilon\to 0} \mathbf{B}^{\mathbf{2},\varepsilon}$

  then one has a so-called geometric rough path above B
  $\hookrightarrow$ easier physical interpretation

Levy area construction for fBm: history

Situation 1: $H > 1/4$

$\hookrightarrow$ 3 possible geometric rough paths constructions for B.

- Malliavin calculus tools (Ferreiro-Utzet)
- Regularization or linearization of the fBm path (Coutin-Qian)
- Regularization and covariance computations (Friz et al)

Situation 2: $d = 1$

$\hookrightarrow$ Then one can take $\mathbf{B}^{\mathbf{2}}_{st} = \frac{(B_t - B_s)^2}{2}$

Situation 3: $H \leq 1/4$, $d > 1$

The constructions by approximation diverge

Existence result by dyadic approximation (Lyons-Victoir)

Recent advances (Unterberger, Nualart-T)

for weakly geometric Levy area construction

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fBm kernel

Recall: B is a d-dimensional fBm, with

$$B_t^i = \int_{\mathbb{R}} K_t(u)\, dW_u^i, \qquad t \geq 0,$$

where W is a d-dimensional Wiener process and

$$\begin{aligned} K_t(u) &\approx (t-u)^{H-\frac{1}{2}} \mathbf{1}_{\{0<u<t\}} \\ \partial_t K_t(u) &\approx (t-u)^{H-\frac{3}{2}} \mathbf{1}_{\{0<u<t\}}. \end{aligned}$$

Heuristics: fBm differential

Formal differential:
we have $B_v^j = \int_0^v K_v(u)\, dW_u^j$ and thus formally for $H > 1/2$

$$\dot{B}_v^j = \int_0^v \partial_v K_v(u)\, dW_u^j$$

Formal definition of the area:
Consider B^i. Then formally

$$\begin{aligned}\int_0^1 B_v^i\, dB_v^j &= \int_0^1 B_v^i \left(\int_0^v \partial_v K_v(u)\, dW_u^j\right) dv \\ &= \int_0^1 \left(\int_u^1 \partial_v K_v(u)\, B_v^i\, dv\right) dW_u^j\end{aligned}$$

This works for $H > 1/2$ since $H - 3/2 > -1$.

Heuristics: fBm differential for $H < 1/2$

Formal definition of the area for $H < 1/2$:
Use the regularity of B^i and write

$$\int_0^1 B_v^i \, dB_v^j = \int_0^1 \left(\int_u^1 \partial_v K_v(u) \, B_v^i \, dv \right) \, dW_u^j$$
$$= \int_0^1 \left(\int_u^1 \partial_v K_v(u) \, \delta B_{uv}^i \, dv \right) dW_u^j$$
$$+ \int_0^1 K_1(u) \, B_u^i \, dW_u^j.$$

Control of singularity: $\partial_v K_v(u) \, \delta B_{uv}^i \approx (v-u)^{H-3/2+H}$
$\hookrightarrow$ Definition works for $2H - 3/2 > -1$, i.e. $H > 1/4$!

Hypothesis: $\int_0^1 B_v^i \, dB_v^j$ well defined as stochastic integral

Outline

Space $\mathcal{H}$

Notation: Let

- $\mathcal{E}$ be the set of step-functions $f : \mathbb{R} \to \mathbb{R}$
- B be a 1-d fBm

Recall:

$$R_H(s,t) = \mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$$

Space $\mathcal{H}$: Closure of $\mathcal{E}$ with respect to the inner product

$$\left\langle \mathbf{1}_{[s_1,s_2]}, \mathbf{1}_{[t_1,t_2]} \right\rangle_{\mathcal{H}} = \mathbf{E}\left[\delta B_{s_1 s_2}\, \delta B_{t_1 t_2}\right] \quad (1)$$

$$= R_H(s_2,t_2) - R_H(s_1,t_2) - R_H(s_2,t_1) + R_H(s_1,t_1)$$

$$\equiv \Delta_{[s_1,s_2]\times[t_1,t_2]} R_H$$

Isonormal process

First chaos of B: We set

- $H_1(B) \equiv$ closure in $L^2(\Omega)$ of linear combinations of δB_{st}

Fundamental isometry: The mapping

$$\mathbf{1}_{[t,t']} \mapsto B_{t'} - B_t$$

can be extended to an isometry between $\mathcal{H}$ and $H_1(B)$
$\hookrightarrow$ We denote this isometry by $\varphi \mapsto B(\varphi)$.

Isonormal process: B can be interpreted as

- A centered Gaussian family $\{B(\varphi);\, \varphi \in \mathcal{H}\}$
- Covariance function given by $\mathbf{E}[B(\varphi_1)B(\varphi_2)] = \langle \varphi_1,\, \varphi_2 \rangle_{\mathcal{H}}$

Underlying Wiener process on compact intervals

Volterra type representation for B:

$$B_t = \int_{\mathbb{R}} K_t(u)\, dW_u, \qquad t \geq 0$$

with

- W Wiener process
- $K_t(u)$ defined by

$$K_t(u) = c_H \Big[\left(\frac{u}{t}\right)^{\frac{1}{2}-H} (t-u)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) u^{\frac{1}{2}-H} \int_u^t v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}}\, dv \Big] \mathbf{1}_{\{0<u<t\}}$$

Bounds on K: If $H < 1/2$

$$|K_t(u)| \lesssim (t-u)^{H-\frac{1}{2}} + u^{H-\frac{1}{2}}, \quad \text{and} \quad |\partial_t K_t(u)| \lesssim (t-u)^{H-\frac{3}{2}}.$$

Underlying Wiener process on $\mathbb{R}$

Mandelbrot's representation for B:

$$B_t = \int_{\mathbb{R}} K_t(u)\, dW_u, \qquad t \geq 0$$

with

- W two-sided Wiener process
- $K_t(u)$ defined by

$$K_t(u) = c_H \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathbf{1}_{\{-\infty < u < t\}}$$

Bounds on K: If $H < 1/2$ and $0 < u < t$

$$|K_t(u)| \lesssim (t-u)^{H-\frac{1}{2}}, \quad \text{and} \quad |\partial_t K_t(u)| \lesssim (t-u)^{H-\frac{3}{2}}.$$

Fractional derivatives

Definition: For $\alpha \in (0,1)$, $u \in \mathbb{R}$ and f smooth enough,

$$\mathcal{D}_-^\alpha f_u = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f_r - f_{u+r}}{r^{1+\alpha}} dr$$

$$\mathcal{I}_-^\alpha f_u = \frac{1}{\Gamma(\alpha)} \int_u^\infty \frac{f_r}{(r-u)^{1-\alpha}} dr$$

Inversion property:

$$\mathcal{I}_-^\alpha \left(\mathcal{D}_-^\alpha f \right) = \mathcal{D}_-^\alpha \left(\mathcal{I}_-^\alpha f \right) = f$$

Fractional derivatives on intervals

Notation: For $f : [a,b] \to \mathbb{R}$, extend f by setting $f^\star = f\,\mathbf{1}_{[a,b]}$

Definition:

$$\mathcal{D}_-^\alpha f_u^\star = \mathcal{D}_{b-}^\alpha f_u = \frac{f_u}{\Gamma(1-\alpha)(b-u)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_u^b \frac{f_u - f_r}{(r-u)^{1+\alpha}} dr$$

$$\mathcal{I}_-^\alpha f_u^\star = \mathcal{I}_{b-}^\alpha f_u = \frac{1}{\Gamma(\alpha)} \int_u^b \frac{f_r}{(r-u)^{1-\alpha}} dr$$

A related operator: For $H < 1/2$,

$$\mathcal{K} f = \mathcal{D}_-^{1/2-H} f$$

Wiener space and fractional derivatives

Proposition 2.

For $H < 1/2$ we have

- $\mathcal{K}$ isometry between $\mathcal{H}$ and a closed subspace of $L^2(\mathbb{R})$
- For $\phi, \psi \in \mathcal{H}$,

$$\mathbf{E}\left[B(\phi)B(\psi)\right] = \langle \phi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{K}\phi, \mathcal{K}\psi \rangle_{L^2(\mathbb{R})},$$

- In particular, for $\phi \in \mathcal{H}$,

$$\mathbf{E}\left[|B(\phi)|^2\right] = \|\varphi\|_{\mathcal{H}} = \|\mathcal{K}\varphi\|_{L^2(\mathbb{R})}$$

Notation:

$B(\phi)$ is called Wiener integral of ϕ w.r.t B

Cylindrical random variables

Definition 3.

Let

- $f \in C_b^\infty(\mathbb{R}^k; \mathbb{R})$
- $\varphi_i \in \mathcal{H}$, for $i \in \{1, \ldots, k\}$
- F a random variable defined by

$$F = f(B(\varphi_1), \ldots, B(\varphi_k))$$

We say that F is a smooth cylindrical random variable

Notation:

$\mathcal{S} \equiv$ Set of smooth cylindrical random variables

Malliavin's derivative operator

Definition for cylindrical random variables:
If $F \in \mathcal{S}$, $DF \in \mathcal{H}$ defined by

$$DF = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_k))\varphi_i .$$

Proposition 4.

D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$.

Notation: $\mathbb{D}^{1,2} \equiv$ closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 = \mathbf{E}\left[|F|^2\right] + \mathbf{E}\left[\|DF\|_{\mathcal{H}}^2\right].$$

Divergence operator

Definition 5.

Domain of definition:

$$\mathrm{Dom}(\delta^\diamond) = \left\{\phi \in L^2(\Omega; \mathcal{H});\ \mathbf{E}\left[\langle DF, \phi\rangle_{\mathcal{H}}\right] \le c_\phi \|F\|_{L^2(\Omega)}\right\}$$

Definition by duality: For $\phi \in \mathrm{Dom}(I)$ and $F \in \mathbb{D}^{1,2}$

$$\mathbf{E}\left[F\,\delta^\diamond(\phi)\right] = \mathbf{E}\left[\langle DF, \phi\rangle_{\mathcal{H}}\right] \qquad (2)$$

Divergence and integrals

Case of a simple process: Consider

- $n \geq 1$
- $0 \leq t_1 < \cdots < t_n$
- $a_i \in \mathbb{R}$ constants

Then

$$\delta^\diamond\left(\sum_{i=0}^{n-1} a_i \mathbf{1}_{[t_i, t_{i+1})}\right) = \sum_{i=0}^{n-1} a_i \, \delta B_{t_i t_{i+1}}$$

Case of a deterministic process: if $\phi \in \mathcal{H}$ is deterministic,

$$\delta^\diamond(\phi) = B(\phi),$$

hence divergence is an extension of Wiener's integral

Divergence and integrals (2)

Proposition 6.

Let

- B a fBm with Hurst parameter $1/4 < H \leq 1/2$
- f a $\mathcal{C}^3$ function with exponential growth
- $\{\Pi_{st}^n;\ n \geq 1\} \equiv$ set of dyadic partitions of $[s, t]$

Define

$$\tilde{S}^{n,\diamond} = \sum_{k=0}^{2^n-1} f(B_{t_k}) \diamond \delta B_{t_k t_{k+1}}.$$

Then $\tilde{S}^{n,\diamond}$ converges in $L^2(\Omega)$ to $\delta^\diamond(f(B))$

Remark: In the Brownian case
$\hookrightarrow$ $\delta^\diamond$ coincides with Itô's integral

Criterion for the definition of divergence

Proposition 7.

Let

- $a < b$, and $\mathcal{E}^{[a,b]} \equiv$ step functions in $[a, b]$
- $\mathcal{H}_0([a, b]) \equiv$ closure of $\mathcal{E}^{[a,b]}$ with respect to

$$\|\varphi\|^2_{\mathcal{H}_0([a,b])} = \int_a^b \frac{\varphi_u^2}{(b-u)^{1-2H}} du + \int_a^b \left(\int_u^b \frac{|\varphi_r - \varphi_u|}{(r-u)^{3/2-H}} dr \right)^2 du.$$

Then

- $\mathcal{H}_0([a, b])$ is continuously included in $\mathcal{H}$
- If $\phi \in \mathbb{D}^{1,2}(\mathcal{H}_0([a, b]))$, then $\phi \in \mathrm{Dom}(\delta^\diamond)$

Bound on the divergence

Corollary 8.

Under the assumptions of Proposition 7,

$$\mathbf{E}\left[\left|\delta^{\diamond}(\phi)\right|^2\right] \lesssim \mathbf{E}\left[\|\phi\|^2_{\mathbb{D}^{1,2}(\mathcal{H}_0([a,b]))}\right]$$

Multidimensional extensions

Aim:
Define a Malliavin calculus for $(B^1, \ldots, B^d)$

First point of view: Rely on

- Partial derivatives D^{B^i} with respect to each component
- Divergences $\delta^{\diamond, B^i}$, defined by duality
  $\hookrightarrow$ Related to integrals with respect to each B^i

Second point of view:
Change the underlying Hilbert space and consider

$$\hat{\mathcal{H}} = \mathcal{H} \times \{1, \ldots, d\}$$

Russo-Vallois' symmetric integral

Definition 9.

Let

- ϕ be a random path

Then

$$\int_a^b \phi_w \, d^\circ B_w^i = L^2 - \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_a^b \phi_w \left(B_{w+\varepsilon}^i - B_{w-\varepsilon}^i \right) dw,$$

provided the limit exists.

Extension of classical integrals: Russo-Vallois' integral coincides with

- Young's integral if $H > 1/2$ and $\phi \in \mathcal{C}^{1-H+\varepsilon}$
- Stratonovich's integral in the Brownian case

Russo-Vallois and Malliavin

Proposition 10.

Let ϕ be a stochastic process such that

1. $\phi \mathbf{1}_{[a,b]} \in \mathbb{D}^{1,2}(\mathcal{H}_0([a,b]))$, for all $-\infty < a < b < \infty$
2. The following is an almost surely finite random variable:

$$\mathrm{Tr}_{[a,b]} D\phi := L^2 - \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_a^b \langle D\phi_u, \mathbf{1}_{[u-\varepsilon,u+\varepsilon]} \rangle_{\mathcal{H}} du$$

Then $\int_a^b \phi_u d^\circ B_u^i$ exists, and verifies

$$\int_a^b \phi_u d^\circ B_u^i = \delta^\diamond(\phi \, \mathbf{1}_{[a,b]}) + \mathrm{Tr}_{[a,b]} D\phi.$$

Outline

Levy area: definition of the divergence

Lemma 11.

Let

- $H > \frac{1}{4}$
- B a d-dimensional fBm(H)
- $0 \leq s < t < \infty$

Then for any $i, j \in \{1, \ldots, d\}$ (either $i = j$ or $i \neq j$) we have

1. $\phi_u^j \equiv \delta B_{su}^j \mathbf{1}_{[s,t]}(u)$ lies in $\text{Dom}(\delta^{\diamond, B^i})$
2. The following estimate holds true:

$$\mathbf{E}\left[\left(\delta^{\diamond, B^i}\left(\phi^j\right)\right)^2\right] \leq c_H |t-s|^{4H}$$

Proof

Case $i = j$, strategy:

- We invoke Corollary 8
- We have to prove $\phi^i \mathbf{1}_{[s,t]} \in \mathbb{D}^{1,2,B^i}(\mathcal{H}_0([s,t]))$
- Abbreviation: we write $\mathbb{D}^{1,2,B^i}(\mathcal{H}_0([s,t])) = \mathbb{D}^{1,2}(\mathcal{H}_0)$

Norm of ϕ^i in $\mathcal{H}_0$: We have

$$\begin{aligned}
\mathbf{E}\left[\|\phi^i\|_{\mathcal{H}_0}^2\right] &= A_{st}^1 + A_{st}^2 \\
A_{st}^1 &= \int_s^t \frac{\mathbf{E}\left[|\delta B_{su}^i|^2\right]}{(t-u)^{1-2H}} du \\
A_{st}^2 &= \mathbf{E}\left\{\int_s^t \left(\int_u^t \frac{|\delta B_{ur}^i|}{(r-u)^{3/2-H}} dr\right)^2 du\right\}
\end{aligned}$$

Proof (2)

Analysis of A^1_{st}:

$$A^1_{st} = \int_s^t \frac{|u-s|^{2H}}{(t-u)^{1-2H}} du \overset{u:=s+(t-s)v}{=} (t-s)^{4H} \int_0^1 \frac{v^{2H}}{(1-v)^{1-2H}} dv$$
$$= c_H (t-s)^{4H}$$

Analysis of A^2_{st}:

$$A^2_{st} = \int_s^t du \int_{[u,t]^2} dr_1 dr_2 \frac{\mathbf{E}\left[\delta B^i_{ur_1} \delta B^i_{ur_2}\right]}{(r_1-u)^{3/2-H}(r_2-u)^{3/2-H}}$$
$$\leq \int_s^t du \left(\int_u^t \frac{dr}{(r-u)^{3/2-2H}}\right)^2$$
$$\leq c_H \int_s^t (t-u)^{4H-1} du = c_H (t-s)^{4H}$$

Proof (3)

Conclusion for $\|\phi^i\|_{\mathcal{H}_0}$: We have found

$$\mathbf{E}\left[\|\phi^i\|_{\mathcal{H}_0}^2\right] \leq c_H (t-s)^{4H}$$

Derivative term, strategy: setting $D = D^{B^i}$ we have

- We have $D_v \phi_u^i = \mathbf{1}_{[s,u]}(v)$
- We have to evaluate $D\phi^i \in \mathcal{H}_0^u \otimes \mathcal{H}^v$

Computation of the $\mathcal{H}$-norm: According to (1),

$$\|D\phi^i\|_{\mathcal{H}}^2 = \mathbf{E}\left[|\delta B_{su}^2|^2\right] = |u-s|^{2H}$$

Proof (4)

Computation for $D\phi^i$: We get

$$\begin{aligned}
\mathbf{E}\left[\|D\phi^i\|^2_{\mathcal{H}_0\otimes\mathcal{H}}\right] &= B^1_{st} + B^2_{st} \\
B^1_{st} &= \int_s^t \frac{\mathbf{E}\left[(u-s)^{2H}\right]}{(t-u)^{1-2H}}\,du \\
B^2_{st} &= \mathbf{E}\left\{\int_s^t \left(\int_u^t \frac{|r-s|^H - |u-s|^H}{(r-u)^{3/2-H}}\,dr\right)^2 du\right\}
\end{aligned}$$

Moreover:

$$0 \le |r-s|^H - |u-s|^H \le |r-u|^H$$

Hence, as for the terms A^1_{st}, A^2_{st}, we get

$$\mathbf{E}\left[\|D\phi^i\|^2_{\mathcal{H}_0\otimes\mathcal{H}}\right] \le c_H (t-s)^{4H}$$

Proof (5)

Summary: We have found

$$\mathbf{E}\left[\|\phi^i\|^2_{\mathcal{H}_0}\right] + \mathbf{E}\left[\|D\phi^i\|^2_{\mathcal{H}_0\otimes\mathcal{H}}\right] \le c_H(t-s)^{4H}$$

Conclusion for B^i: According to Proposition 7 and Corollary 8

- $\delta B^i_{s\cdot}\mathbf{1}_{[s,t]} \in \mathrm{Dom}(\delta^{\diamond,B^i})$
- We have

$$\mathbf{E}\left[\left(\delta^{\diamond,B^i}\left(\delta B^i_{s\cdot}\mathbf{1}_{[s,t]}\right)\right)^2\right] \le c_H|t-s|^{4H}$$

Proof (6)

Case $i \neq j$, strategy: Conditioned on $\mathcal{F}^{B^j}$

- B^j and $\phi^j = \delta B^j_{s\cdot}$ are deterministic
- $\delta^{\diamond, B^i}(\phi^j)$ is a Wiener integral

Computation: For $i \neq j$ we have

$$\begin{aligned} \mathbf{E}\left[\left(\delta^{\diamond,B^i}\left(\phi^j\right)\right)^2\right] &= \mathbf{E}\left\{\mathbf{E}\left[\left(\delta^{\diamond,B^i}\left(\phi^j\right)\right)^2 \,\middle|\, \mathcal{F}^{B^j}\right]\right\} \\ &= \mathbf{E}\left[\|\phi^j\|^2_{\mathcal{H}}\right] \\ &\leq c_H \mathbf{E}\left[\|\phi^j\|^2_{\mathcal{H}_0}\right] \\ &\leq c_H |t-s|^{4H}, \end{aligned} \tag{3}$$

where computations for the last step are the same as for $i = j$.

Definition of the Levy area

Proposition 12.

Let

- $H > \frac{1}{4}$
- B a d-dimensional fBm(H)
- $0 \leq s < t < \infty$

Then for any $i, j \in \{1, \ldots, d\}$ (either $i = j$ or $i \neq j$) we have

1. $\mathbf{B}^{\mathbf{2},ji}_{st} \equiv \int_s^t \delta B^j_{su} \, d^\circ B^i_u$ defined in the Russo-Vallois sense
2. The following estimate holds true:

$$\mathbf{E}\left[\left|\mathbf{B}^{\mathbf{2},ji}_{st}\right|^2\right] \leq c_H |t-s|^{4H}$$

Proof

Strategy:

- We apply Proposition 10, and check the assumptions
- Proposition 10, item 1: proved in Lemma 11
- Proposition 10, item 2: need to compute trace term

Trace term, case $i = j$: We have

$$D_v^{B^i} \phi_u^i = \mathbf{1}_{[s,u]}(v)$$

Hence

$$\langle D\phi_u^i, \mathbf{1}_{[u-\varepsilon,u+\varepsilon]} \rangle_{\mathcal{H}} = \Delta_{[s,u]\times[u-\varepsilon,u+\varepsilon]} R_H$$

Proof (2)

Computation of the rectangular increment: We have

$$
\begin{aligned}
&\Delta_{[s,u]\times[u-\varepsilon,u+\varepsilon]} R_H \\
&= R_H(u, u+\varepsilon) - R_H(s, u+\varepsilon) - R_H(u, u-\varepsilon) + R_H(s, u-\varepsilon) \\
&= \frac{1}{2}\left[-\varepsilon^{2H} + (u-s+\varepsilon)^{2H} + \varepsilon^{2H} - (u-s-\varepsilon)^{2H}\right] \\
&= \frac{1}{2}\left[(u-s+\varepsilon)^{2H} - (u-s-\varepsilon)^{2H}\right]
\end{aligned}
$$

Computation of the integral: Thanks to an elementary integration,

$$
\begin{aligned}
&\int_s^t \Delta_{[s,u]\times[u-\varepsilon,u+\varepsilon]} R_H \, du \\
&= \frac{1}{2(2H+1)}\left[(t-s+\varepsilon)^{2H+1} - \varepsilon^{2H+1} - (t-s-\varepsilon)^{2H+1}\right]
\end{aligned}
$$

Proof (3)

Computation of the trace term: Differentiating we get

$$\begin{aligned}
&\mathrm{Tr}_{[s,t]} D\phi^i \\
&= \frac{1}{2(2H+1)} \lim_{\varepsilon \to 0} \frac{(t-s+\varepsilon)^{2H+1} - \varepsilon^{2H+1} - (t-s-\varepsilon)^{2H+1}}{2\varepsilon} \\
&= \frac{(t-s)^{2H}}{2}
\end{aligned}$$

Expression for the Stratonovich integral: According to Proposition 10

$$\mathbf{B}^{\mathbf{2},ii}_{st} = \int_s^t \delta^i_{su} d^\circ B^i_u = \delta^{\diamond, B^i}(\phi^i \mathbf{1}_{[s,t]}) + \frac{(t-s)^{2H}}{2} \tag{4}$$

Proof (4)

Moment estimate: Thanks to relation (4) we have

$$\mathbf{E}\left[\left|\mathbf{B}^{\mathbf{2},ii}_{st}\right|^2\right] \leq c_H |t-s|^{4H}$$

Case $i \neq j$: We have

- Trace term is 0
- $\mathbf{B}^{\mathbf{2},ji}_{st} = \delta^{\diamond,B^i}(\phi^j \mathbf{1}_{[s,t]})$
- Moment estimate follows from Lemma 11

Remark

Another expression for $\mathbf{B}^{ii}$:

Rules of Stratonovich calculus for B show that

$$\mathbf{B}_{st}^{ii} = \frac{(\delta B_{st}^{i})^2}{2}$$

Much simpler expression!

Outline

Levy area construction

Summary: for $0 \leq s < t \leq \tau$, we have defined the stochastic integral

$$\mathbf{B}^{\mathbf{2}}_{st} = \int_s^t \int_s^u d^\circ B_v \otimes d^\circ B_u, \quad \text{i. e.} \quad \mathbf{B}^{\mathbf{2},ij}_{st} = \int_s^t \int_s^u d^\circ B^i_v \, d^\circ B^j_u,$$

If $i = j$:

- $\mathbf{B}^{\mathbf{2}}_{st}(i,i) = \frac{1}{2}(B_t - B_s)^2$

If $i \neq j$:

- B^i considered as deterministic path
- $\mathbf{B}^{\mathbf{2},ij}_{st}$ is a Wiener integral w.r.t B^j

Algebraic relation

Proposition 13.

Let

- $(s, u, t) \in \mathcal{S}_{3,\tau}$
- $\mathbf{B}^2$ as constructed in Proposition 12

Then we have

$$\delta \mathbf{B}^{2,ij}_{sut} = \delta B^i_{su} \, \delta B^j_{ut}$$

Proof

Levy area as a limit: from definition of R-V integral we have

$$\mathbf{B}^{\mathbf{2},ij}_{st} = \lim_{\varepsilon\to 0} \mathbf{B}^{\mathbf{2},\varepsilon,ij}_{st}, \quad \text{where} \quad \mathbf{B}^{\mathbf{2},\varepsilon,ij}_{st} = \int_s^t \delta B^i_{sv}\, dX^{\varepsilon,j}_v,$$

with

$$X^{\varepsilon,j}_v = \int_0^v \frac{1}{2\varepsilon}\, \delta B^j_{w-\varepsilon,w+\varepsilon}\, dw$$

Increments of $\mathbf{B}^{\mathbf{2},\varepsilon,ij}$: $\mathbf{B}^{\mathbf{2},\varepsilon,ij}_{st}$ is a Riemann type integral and

$$\delta\mathbf{B}^{\mathbf{2},\varepsilon,ij}_{sut} = \delta B^i_{su}\, \delta X^{\varepsilon,j}_{ut} \tag{5}$$

We wish to take limits in (5)

Proof (2)

Limit in the lhs of (5): We have seen

$$\lim_{\varepsilon\to 0} \delta \mathbf{B}^{\mathbf{2},\varepsilon,ij}_{sut} \overset{L^2(\Omega)}{=} \delta \mathbf{B}^{\mathbf{2},ij}_{sut}$$

Expression for $X^{\varepsilon,j}$: We have

$$\begin{aligned} X^{\varepsilon,j}_v &= \frac{1}{2\varepsilon}\left\{\int_0^v B^j_{w+\varepsilon}\,dw - \int_0^v B^j_{w-\varepsilon}\,dw\right\} \\ &= \frac{1}{2\varepsilon}\left\{\int_\varepsilon^{v+\varepsilon} B^j_w\,dw - \int_{-\varepsilon}^{v-\varepsilon} B^j_w\,dw\right\} \\ &= \frac{1}{2\varepsilon}\left\{\int_{v-\varepsilon}^{v+\varepsilon} B^j_w\,dw - \int_{-\varepsilon}^{\varepsilon} B^j_w\,dw\right\} \end{aligned} \tag{6}$$

Proof (3)

Limit in the rhs of (5):
Invoking Lebesgue's differentiation theorem in (6), we get

$$\lim_{\varepsilon \to 0} X_v^{\varepsilon,j} = \delta B_{0v}^j = B_v^j \quad \Longrightarrow \quad \lim_{\varepsilon \to 0} \delta B_{su}^i \, \delta X_{ut}^{\varepsilon,j} = \delta B_{su}^i \, \delta B_{ut}^j$$

Conclusion: Taking limits on both sides of (5), we get

$$\delta \mathbf{B}_{sut}^{\mathbf{2},ij} = \delta B_{su}^i \, \delta B_{ut}^j$$

Regularity criterion in $\mathcal{C}_2$

Lemma 14.

Let $g \in \mathcal{C}_2$. Then, for any $\gamma > 0$ and $p \geq 1$ we have

$$\|g\|_\gamma \leq c \left(U_{\gamma;p}(g) + \|\delta g\|_\gamma \right),$$

with

$$U_{\gamma;p}(g) = \left(\int_0^T \int_0^T \frac{|g_{st}|^p}{|t-s|^{\gamma p+2}} ds\, dt \right)^{1/p}.$$

Levy area of fBm: regularity

Proposition 15.

Let

- $\mathbf{B}^2$ as constructed in Proposition 12
- $0 < \gamma < H$

Then, up to a modification, we have

$$\mathbf{B}^2 \in \mathcal{C}_2^{2\gamma}([0,\tau];\mathbb{R}^{d,d})$$

Proof

Strategy: Apply our regularity criterion to $g = \mathbf{B}^2$

Term 2: We have seen: $\delta\mathbf{B}^2 = \delta B \otimes \delta B$

$$B \in \mathcal{C}_1^{\gamma} \quad \Rightarrow \quad \delta B \otimes \delta B \in \mathcal{C}_3^{2\gamma}$$

Term 1: For $p \geq 1$ we shall control

$$E\left[\left|U_{\gamma;p}(\mathbf{B}^2)\right|^p\right] = \int_0^T \int_0^T \frac{\mathbf{E}\left[|\mathbf{B}^2_{st}|^p\right]}{|t-s|^{\gamma p}} ds\, dt$$

Proof (2)

Aim: Control of $\mathbf{E}\left[|\mathbf{B}^{\mathbf{2}}_{st}|^p\right]$

Scaling and stationarity arguments:

$$\begin{aligned}\mathbf{E}\left[|\mathbf{B}^{\mathbf{2},ij}_{st}|^p\right] &= \mathbf{E}\left[\left|\int_s^t dB^i_u \int_s^u dB^j_v\right|^p\right] \\ &= |t-s|^{2pH}\,\mathbf{E}\left[\left|\int_0^1 dB^i_u \int_0^u dB^j_v\right|^p\right]\end{aligned}$$

Stochastic analysis arguments:

Since $\int_0^1 dB^i_u \int_0^u dB^j_v$ is element of the second chaos of fBm:

$$\mathbf{E}\left[\left|\int_0^1 dB^i_u \int_0^u dB^j_v\right|^p\right] \le c_{p,1}\,\mathbf{E}\left[\left|\int_0^1 dB^i_u \int_0^u dB^j_v\right|^2\right] \le c_{p,2}$$

Proof (3)

Recall: $\|\mathbf{B^2}\|_\gamma \leq c \left(U_{\gamma;p}(\mathbf{B^2}) + \|\delta\mathbf{B^2}\|_\gamma \right)$

Computations for $U_{\gamma;p}(\mathbf{B^2})$:
Let $\gamma < 2H$, and p such that $\gamma + 2/p < 2H$. Then:

$$E\left[\left| U_{\gamma;p}(\mathbf{B^2}) \right|^p \right] = \int_0^T \int_0^T \frac{E\left[|\mathbf{B^2_{st}}|^p \right]}{|t-s|^{\gamma p + 2}} ds\, dt$$

$$\leq c_p \int_0^T \int_0^T \frac{|t-s|^{2pH}}{|t-s|^{p(\gamma+2/p)}} ds\, dt \leq c_p$$

Conclusion:

- $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}$ for any $\gamma < H$
- One can solve RDEs driven by fBm with $H > 1/3$!

Outline

p-variation in $\mathbb{R}^2$

Definition 16.

Let

- X centered Gaussian process on $[0, T]$
- $R : [0, T]^2 \to \mathbb{R}$ covariance function of X
- $0 \leq s < t \leq T$
- $\Pi_{st} \equiv$ set of partitions of $[s, t]$

We set

$$\|R\|_{p-\text{var};\,[s,t]^2}^p = \sup_{\Pi_{st}^2} \sum_{i,j} \left| \Delta_{[s_i, s_{i+1}]\times[t_j, t_{j+1}]} R \right|^p$$

and

$$\mathcal{C}^{p-\text{var}} = \left\{ f : [0, T]^2 \to \mathbb{R};\ \|R\|_{p-\text{var}} < \infty \right\}$$

Young's integral in the plane

Proposition 17.

Let

- $f \in \mathcal{C}^{p-\text{var}}$
- $g \in \mathcal{C}^{q-\text{var}}$
- p, q such that $\frac{1}{p} + \frac{1}{q} > 1$

Then the following integral is defined in Young's sense:

$$\int_{[s,t]^2} \Delta_{[s,u_1]\times[s,u_2]} f \, dg(u_1, u_2)$$

Area and 2d integrals

Proposition 18.

Let

- $X \in \mathbb{R}^d$ smooth centered Gaussian process on $[0, T]$
- Independent components X^j
- $R : [0, T]^2 \to \mathbb{R}$ common covariance function of X^j's
- $0 \le s < t \le T$ and $i \ne j$

Define (in the Riemann sense) $\mathbf{X}^{\mathbf{2},ij}_{st} = \int_s^t \delta X^i_{su} dX^j_u$. Then

$$\mathbf{E}\left[\left|\mathbf{X}^{\mathbf{2},ij}_{st}\right|^2\right] = \int_{[s,t]^2} \Delta_{[s,u_1]\times[s,u_2]} R \, dR(u_1, u_2) \qquad (7)$$

Proof

Expression for the area: We have

$$\mathbf{X}^{\mathbf{2},ij}_{st} = \int_s^t \delta X^i_{su} dX^j_u = \int_s^t \delta X^i_{su}\, \dot{X}^j_u\, du$$

Expression for the second moment:

$$\begin{aligned}
\mathbf{E}\left[\left|\mathbf{X}^{\mathbf{2},ij}_{st}\right|^2\right] &= \int_{[s,t]^2} \mathbf{E}\left[\delta X^i_{su_1} \delta X^i_{su_2} \dot{X}^j_{u_1} \dot{X}^j_{u_2}\right]\, du_1 du_2 \\
&= \int_{[s,t]^2} \mathbf{E}\left[\delta X^i_{su_1} \delta X^i_{su_2}\right] \mathbf{E}\left[\dot{X}^j_{u_1} \dot{X}^j_{u_2}\right]\, du_1 du_2 \\
&= \int_{[s,t]^2} \Delta_{[s,u_1]\times[s,u_2]} R\ \partial^2_{u_1 u_2} R(u_1, u_2)\, du_1 du_2 \\
&= \int_{[s,t]^2} \Delta_{[s,u_1]\times[s,u_2]} R\, dR(u_1, u_2)
\end{aligned}$$

Remarks

Expression in terms of norms in $\mathcal{H}$: We also have

$$\begin{aligned}\mathbf{E}\left[\left|\mathbf{X}^{\mathbf{2},ij}_{st}\right|^2\right] &= \int_{[s,t]^2} \mathbf{E}\left[\delta B^i_{su_1}\delta B^i_{su_2}\right]\, dR(u_1,u_2) \\ &= \mathbf{E}\left[\left\langle \delta B^i_{s\cdot},\, \delta B^i_{s\cdot}\right\rangle_{\mathcal{H}}\right]\end{aligned}$$

This is similar to (3)

Extension:

- Formula (7) makes sense as long as $R \in \mathcal{C}^{p-\text{var}}$ with $p < 2$
- We will check this assumption for a fBm with $H > \frac{1}{4}$

p-variation of the fBm covariance

Proposition 19.

Let

- B a 1-d fBm with $H < \frac{1}{2}$
- $R \equiv$ covariance function of B
- $T > 0$

Then

$$R \in \mathcal{C}^{\frac{1}{2H}-\text{var}}$$

Proof

Setting: Let

- $0 \leq s < t \leq T$
- $\pi = \{t_i\} \in \Pi_{st}$
- $S_\pi = \sum_{i,j} \left| \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{t_j t_{j+1}} \right] \right|^{\frac{1}{2H}}$
- For a fixed i, $S_\pi^i = \sum_j \left| \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{t_j t_{j+1}} \right] \right|^{\frac{1}{2H}}$

Decomposition: We have

$$S_\pi^i = S_\pi^{i,1} + S_\pi^{i,2},$$

with

$$S_\pi^{i,1} = \sum_{j \neq i} \left| \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{t_j t_{j+1}} \right] \right|^{\frac{1}{2H}}, \quad \text{and} \quad S_\pi^{i,2} = \left| \mathbf{E}\left[\left(\delta B_{t_i t_{i+1}} \right)^2 \right] \right|^{\frac{1}{2H}}$$

Proof (2)

A deterministic bound: If $y_j < 0$ for all $j \neq i$ then

$$\sum_{j\neq i} |y_j|^{\frac{1}{2H}} \leq \left| \sum_{j\neq i} |y_j| \right|^{\frac{1}{2H}} = \left| \sum_{j\neq i} y_j \right|^{\frac{1}{2H}}$$

This applies to $y_j = \mathbf{E}[\delta B_{t_i t_{i+1}} \delta B_{t_i t_{i+1}}]$ when $H < \frac{1}{2}$

Bound for $S_\pi^{i,1}$: Write

$$\begin{aligned} S_\pi^{i,1} &\leq \left| \sum_{j\neq i} \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{t_j t_{j+1}} \right] \right|^{\frac{1}{2H}} \\ &\leq \left| \sum_{j} \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{t_j t_{j+1}} \right] \right|^{\frac{1}{2H}} + \left| \mathbf{E}\left[\left(\delta B_{t_i t_{i+1}} \right)^2 \right] \right|^{\frac{1}{2H}} \\ &= \left| \mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{st} \right] \right|^{\frac{1}{2H}} + \left| \mathbf{E}\left[\left(\delta B_{t_i t_{i+1}} \right)^2 \right] \right|^{\frac{1}{2H}} \end{aligned}$$

Proof (3)

Bound for S_π^i: We have found

$$
\begin{aligned}
S_\pi^i &\leq \left|\mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{st}\right]\right|^{\frac{1}{2H}} + 2\left|\mathbf{E}\left[\left(\delta B_{t_i t_{i+1}}\right)^2\right]\right|^{\frac{1}{2H}} \\
&= \left|\mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{st}\right]\right|^{\frac{1}{2H}} + 2(t_{i+1} - t_i)
\end{aligned}
$$

Bound on increments of R: Let $[u, v] \subset [s, t]$. Then

$$
\begin{aligned}
&\left|\mathbf{E}\left[\delta B_{uv} \delta B_{st}\right]\right| = |R(v,t) - R(u,t) - R(v,s) + R(u,s)| \\
&= \left|(t-v)^{2H} - (t-u)^{2H} - (v-s)^{2H} + (u-s)^{2H}\right| \\
&\leq \left|(t-v)^{2H} - (t-u)^{2H}\right| + \left|(v-s)^{2H} - (u-s)^{2H}\right| \\
&\leq 2(v-u)^{2H}
\end{aligned}
$$

Proof (4)

Bound for S_π^i, ctd: Applying the previous estimate,

$$\begin{aligned} S_\pi^i &\leq \left|\mathbf{E}\left[\delta B_{t_i t_{i+1}} \delta B_{st}\right]\right|^{\frac{1}{2H}} + 2(t_{i+1} - t_i) \\ &\leq 4(t_{i+1} - t_i) \end{aligned}$$

Bound for S_π: We have

$$S_\pi \leq \sum_i S_\pi^i \leq 4(t-s)$$

Conclusion:

Since π is arbitrary, we get the finite $\frac{1}{2H}$-variation

Construction of the Levy area

Strategy:

1. Regularize B as B^{ε}
2. For B^{ε}, the previous estimates hold true
3. Then we take limits
   $\hookrightarrow$ This uses the $\frac{1}{2H}$-variation bound, plus rate of convergence
   $\hookrightarrow$ Long additional computations

Outline

Current research directions

Non exhaustive list:

- Further study of the law of Gaussian SDEs:
  Gaussian bounds, hypoelliptic cases
- Ergodicity for rough differential equations
- Statistical aspects of rough differential equations
- New formulations for rough PDEs:
  - Weak formulation (example of conservation laws)
  - Krylov's formulation
- Links between pathwise and probabilistic approaches for SPDEs
  $\hookrightarrow$ Example of PAM in $\mathbb{R}^2$