Inference for the jump part of quadratic variation of Itô semimartingales

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Introduction

General framework

Model: Continuous–time stochastic processes (Itô semimartingales) which allow for stochastic volatility, jumps and leverage effect;

Aims: Learning about volatility and testing for jumps;

Methodology: Non–parametric;

Impacts: Risk management, portfolio selection, option pricing.

Aim of the study

Aim: Inference on jump part of quadratic variation.

Implications: Jump tests.

Outline

Model definition

Estimation and inference for quadratic variation

Realised variance and realised multipower variation

A bivariate central limit theorem

A feasible version of the central limit theorem

Simulation and empirical study

Summary and future work

Model assumptions

The log-price $X = (X_t)_{t \geq 0}$ is an Itô semimartingale on a probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t,$$

where

- W is a Brownian motion,
- J is a jump process satisfying some weak regularity assumptions,
- b is predictable, and
- σ is càdlàg and satisfies some weak regularity assumptions.

Discrete returns and realised variance

Discrete returns

Assume that we observe the process X over an interval $[0, t]$ at times $i\Delta_n$ for $\Delta_n > 0$ and $i = 1, \ldots, [t/\Delta_n]$. So for its discretely observed increments we write

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}.$$

Realised variance

The *realised variance* (RV) is defined by

$$RV_t^n = \sum_{i=1}^{[t/\Delta_n]} \left(\Delta_i^n X\right)^2 .$$

Realised variance and quadratic variation

- RV estimates quadratic variation consistently, i.e.

$$RV_t^n = \sum_{i=1}^{[t/\Delta_n]} \left(\Delta_i^n X\right)^2 \xrightarrow{ucp} [X]_t, \quad \text{as } n \to \infty,$$

  where the convergence is uniformly on compacts in probability (ucp).
- Since $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t$, we have

$$[X]_t = \int_0^t \sigma_s^2 ds + \sum_{0 \le s \le t} (\Delta J_s)^2 .$$

- Aim: Estimation and inference on the jump part of QV:

$$\sum_{0 \le s \le t} (\Delta J_s)^2 .$$

Realised bipower variation

Barndorff-Nielsen & Shephard (2006) showed that *realised bipower variation* is a consistent estimator of the continuous part of the quadratic variation, i.e.

$$\mu_1^{-2} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X||\Delta_{i+1}^n X| \xrightarrow{ucp} \int_0^t \sigma_s^2 ds, \quad \text{as } n \to \infty,$$

where $\mu_r = \mathbb{E}(|U|^r)$ for $r > 0$, $U \sim N(0,1)$. Jacod (2006) proved the robustness towards jumps.

Realised multipower variation

Barndorff-Nielsen & Shephard (2006), Barndorff-Nielsen et al. (2006), Woerner (2006), Jacod (2006) studied *realised multipower variation*: Let $I \geq 2$ denote an integer and

$$RMPV(2; I)_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I} \mu_{2/I}^{-I} \sum_{i=1}^{[t/\Delta_n]-I} \prod_{j=1}^{I} |\Delta_{i+j-1}^n X|^{2/I}.$$

Then

$$RMPV(2; I)_t^n \xrightarrow{ucp} \int_0^t \sigma_s^2 ds, \quad \text{as } n \to \infty.$$

A consistent estimator for the jump part of quadratic variation

Clearly, for an integer $l \geq 2$, we get, as $n \to \infty$:

- *Linear test statistic*

$$RV_t^n - RMPV(2; l)_t^n \xrightarrow{ucp} [Y]_t^d = \sum_{0 \leq s \leq t} (\Delta J_s)^2 \,.$$

- *Ratio test statistic*

$$\frac{RMPV(2; l)_t^n}{RV_t^n} - 1 \xrightarrow{ucp} -\frac{[Y]_t^c}{[Y]_t}$$

(Barndorff-Nielsen & Shephard (2006)).

Towards a central limit theorem: The concept of stable convergence

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote a probability space endowed with a sequence X_n of random variables taking their values in a Polish space $(U, \mathcal{U})$.
- If there is a probability measure μ defined on the extended space $(\Omega \times U, \mathcal{A} \otimes \mathcal{U})$ such that for every bounded $\mathcal{A}$-measurable random variable Z and for every bounded and continuous function g on U we have

$$\mathbb{E}\left(Zg(X_n)\right) \to \int Z(\omega)g(x)\mu(d\omega, dx), \quad \text{as } n \to \infty,$$

then we say that X_n converges *stably in law*. (See e.g. Jacod & Shiryaev (2003)).

A central limit theorem for realised variance

Barndorff-Nielsen and Shephard (2002) proved a central limit theorem in the absence of jumps. Jacod (2007) generalised this result by proving that, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \left(RV_t^n - [X]_{\Delta_n [t/\Delta_n]} \right) \longrightarrow Z_t + Y_t, \tag{1}$$

where the convergence is stably in law as a process and

$$Y_t = \sqrt{2} \int_0^t \sigma_u^2 d\overline{W}_u, \tag{2}$$

$$Z_t = 2 \sum_{p:\ T_p \le t} \Delta X_{T_p} \left(\sigma_{T_p -} \sqrt{\xi_p}\, U_p + \sigma_{T_p} \sqrt{1 - \xi_p}\, U'_p \right), \tag{3}$$

for a Brownian motion $\overline{W}$, $U, U' \sim N(0,1)$ and $\xi \sim U[0,1]$ (all independent) and (T_p) is a sequence of stopping times increasing to $+\infty$.

A central limit theorem for realised multipower variation

Barndorff-Nielsen & Shephard (2006) proved a CLT in the absence of jumps. Woerner (2006) and Jacod (2006) derived a CLT in the presence of jumps:
We need a stronger assumption on the jumps of X: Let β denote the generalised Blumenthal-Getoor of X. Assume that $\beta < 1$ and $\frac{\beta}{2-\beta} < \frac{2}{I} < 1$. Then, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \left(RMPV(2; I)_t^n - [X]_t^c \right) \longrightarrow \widetilde{Y}_t, \tag{4}$$

where the convergence is stably in law as a process and

$$\widetilde{Y}_t = \omega_I \mu_{I/2}^{-I} \int_0^t \sigma_u^2 d\widetilde{W}_u, \tag{5}$$

for an independent Brownian motion $\widetilde{W}$ and a known constant ω_I.

Main result: A bivariate central limit theorem

Let β be the BG index of X. Assume that $\beta < 1$, $\sigma_t > 0 \ \forall t$, and let $\frac{\beta}{2-\beta} < \frac{2}{l} < 1$. Then

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [X]_{\Delta_n[t/\Delta_n]} \\ RMPV(2;l)_t^n - \int_0^t \sigma_u^2 du \end{pmatrix} \xrightarrow{\text{stably in law}}$$

$$\begin{pmatrix} \sqrt{2} \int_0^t \sigma_u^2 d\overline{W}_u + 2 \sum_{p:\ T_p \le t} \Delta X_{T_p} \left(\sqrt{\xi_p} U_p \sigma_{T_{p-}} + \sqrt{1-\xi_p}\ U'_p \sigma_{T_p} \right) \\ \sqrt{2} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\overline{W}_u + \sqrt{\theta_{\mathbf{r}}} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{W}_u \end{pmatrix}$$

where the convergence is stable in law as a process and $\theta_{\mathbf{r}} = (\mu_{\mathbf{r}}^{-1} \sqrt{A(\mathbf{r})})^2 - 2$.
If σ and X do not jump together, the first component is the sum of two independent martingales which have (conditional on $\mathcal{A}$) Gaussian law. Note that in that case $\sigma_{T_{p-}} = \sigma_{T_p}$ since T_p are the jump times of X.

Corollary

Let β be the BG index of X. Assume that $\beta < 1$, $\sigma_t > 0\ \forall t$, and let $\frac{\beta}{2-\beta} < \frac{2}{l} < 1$, and assume that X and σ have no common jumps. For $l \in \mathbb{N}$ with $2 < l < \frac{2}{\beta}(2-\beta)$ we obtain:

$$\frac{1}{\sqrt{\Delta_n}}\left(RV_t^n - RMPV(2;l)_t^n - [X]^d_{\Delta_n[t/\Delta_n]}\right) \xrightarrow{\text{stably in law}} L_t, \qquad (6)$$

where L_t has (conditionally on $\mathcal{A}$), Gaussian law with zero mean and variance given by

$$\theta_l \int_0^t \sigma_u^4 du + 4 \sum_{p:\ T_P \leq t} \sigma_{T_p}^2 \left(\Delta X_{T_p}\right)^2 .$$

Note: This central limit theorem is *infeasible*.

A consistent estimator for the continuous part of the asymptotic variance

Recall that

$$RMPV(4;\widetilde{I}) = \frac{[t/\Delta_n]}{[t/\Delta_n]-\widetilde{I}}\mu_{4/\widetilde{I}}^{-\widetilde{I}} \sum_{i=1}^{[t/\Delta_n]-\widetilde{I}} \prod_{j=1}^{\widetilde{I}} |\Delta_{i+j-1}^n X|^{4/\widetilde{I}}$$

is a consistent estimator of $\int_0^t \sigma_s^4 ds$ in the presence of jumps of X when $\widetilde{I} \geq 3$. Hence,

$$\theta_I RMPV(4;\widetilde{I}) \xrightarrow{ucp} \theta_I \int_0^t \sigma_s^4 ds, \quad \text{as } n \to \infty.$$

A consistent estimator for the jump part of the asymptotic variance

In the following we always assume that X and σ have no common jumps.
We want to estimate:

$$4 \sum_{p:\, T_p \le t} \sigma^2_{T_p} \left(\Delta X_{T_p}\right)^2 .$$

We replace σ^2 by $\widehat{\sigma}^2$ and show that

$$4 \sum_{i=1}^{[t/\Delta_n]} \widehat{\sigma}^2_{(i-1)\Delta_n} \left(\Delta_i^n X\right)^2 \xrightarrow{\mathbb{P}} 4 \sum_{p:\, T_p \le t} \sigma^2_{T_p} \left(\Delta X_{T_p}\right)^2 , \quad \text{as } n \to \infty.$$

A consistent estimator for the spot variance

- E. g. one can use the local volatility estimator based on locally averaged realised bipower variation or locally averaged truncated realised variance.
- Let $K_n > 0$ such that $K_n \to \infty$ and $K_n \Delta_n \to 0$ as $n \to \infty$.
- Locally averaged realised bipower variation:

$$\hat{\sigma}^2_{(i-1)\Delta_n} = \frac{1}{K_n - 2} \sum_{j=i-K_n+2}^{i-1} \left|\Delta_j^n X\right| \left|\Delta_{j-1}^n X\right|,$$

  as studied by Lee & Mykland (2006).
- Locally averaged truncated realised variance:

$$\frac{1}{K_n} \sum_{j=i-K_n}^{i-1} \left(\Delta_j^n X\right)^2 \mathbb{1}_{\{|\Delta_i^n X| \leq \alpha \Delta_n^\omega\}},$$

  where $\alpha > 0$ and $\omega \in (0, 1/2)$ as studied by Ait-Sahalia & Jacod (2006)

Results: Consistent estimators for the asymptotic variance

Lemma

$$\sum_{i=1}^{[t/n]} \hat{\sigma}^2_{(i-1)\Delta_n} \left(\Delta_i^n X\right)^2 \xrightarrow{\mathbb{P}} \int_0^t \sigma_s^2 d[X]_s, \quad \text{as } n \to \infty, \qquad (7)$$

where

$$\int_0^t \sigma_s^2 d[X]_s = \int_0^t \sigma_s^4 ds + \sum_{0 \le s \le t} \sigma_s^2 \left(\Delta X_s\right)^2 .$$

Hence we get

$$\sum_{i=1}^{[t/n]} \hat{\sigma}^2_{(i-1)\Delta_n} \left(\Delta_i^n X\right)^2 - RMPV(4; \tilde{l}) \xrightarrow{\mathbb{P}} \sum_{0 \le s \le t} \sigma_s^2 \left(\Delta X_s\right)^2$$

Finite sample correction

Since the estimator above can be negative in finite samples we choose $\widehat{\Sigma}_t^n = \widehat{\Sigma}_t^n(l, \widetilde{l})$ for integers $l, \widetilde{l} \geq 3$ and

$$\widehat{\Sigma}_t^n = \max\left\{4 \sum_{i=1}^{[t/\Delta_n]} \hat{\sigma}^2_{(i-1)\Delta_n} \left(\Delta_i^n X\right)^2 \right.$$
$$\left. -(4-\theta_l) RMPV(4;\widetilde{l}),\ \theta_l RMPV(4;\widetilde{l})\right\} \quad (8)$$

as estimator for the asymptotic variance.
Hence,

$$\widehat{\Sigma}_t^n \xrightarrow{\mathbb{P}} \theta_l \int_0^t \sigma_s^4 ds + 4 \sum_{0 \leq s \leq t} \sigma_s^2 (\Delta X_s)^2, \qquad \text{as } n \to \infty.$$

A feasible central limit theorem

Let β be the Blumenthal–Getoor index of X. Assume that $\beta < 1$, $\sigma_t > 0\ \forall t$, and let $\frac{\beta}{2-\beta} < \frac{2}{l} < 1$, and assume that X and σ have no common jumps. For $l \in \mathbb{N}$ with $2 < l < \frac{2}{s}(2-s)$ we obtain:

$$\frac{\left(RV_t^n - RMPV(2;l)_t^n - [X]^d_{\Delta_n[t/\Delta_n]}\right)}{\sqrt{\Delta_n}\widehat{\Sigma}_t^n} \overset{stably\ \ in\ \ law}{\longrightarrow} N(0,1),$$

as $n \to \infty$.

A quick look at some of the simulation results I

Stochastic volatility jump diffusion

$$dX_t = \mu dt + \exp(\beta_0 + \beta_1 v_t) dW_t^X + dL_t^J,$$
$$dv_t = \alpha_v v_t dt + dW_t^v,$$

where W^X, W^v are standard Brownian motions with $Corr(dW^X, dW^v) = \rho$, v_t is the stochastic volatility factor, L_t^J compound Poisson process with constant jump intensity λ and jump size distribution $N(0,\ \sigma_{jmp}^2)$. Choice of parameters: $\mu = 0.03$, $\beta_0 = 0$, $\beta_1 = 0.125$, $\rho = -0.62$, $\alpha_v = -0.1$, $\lambda = 0.118$, $\sigma_{jmp} = 1.5$ (see Huang & Tauchen (2005)).

A quick look at some of the simulation results II

		Linear test		
$[1/\Delta_n]$ **(K_n)**	**I**	**Mean**	**S.D.**	**Cove.**
39	3	-0.09	0.96	0.966
(7)	4	-0.05	0.98	0.966
	10	0	1.11	0.927
78	3	-0.08	0.93	0.969
(9)	4	-0.05	0.95	0.966
	10	0	1.02	0.945
390	3	-0.05	0.96	0.962
(20)	4	-0.02	0.96	0.963
	10	0.01	0.98	0.958
1560	3	-0.05	0.96	0.957
(40)	4	-0.02	0.96	0.958
	10	-0.01	0.98	0.952
23400	3	-0.02	0.99	0.953
(153)	4	0	0.99	0.953
	10	0	0.98	0.954

Table: Simulation results for the stochastic volatility jump diffusion model. We simulate data for 5000 days and compute the mean, standard deviation and coverage of the feasible linear test statistic.

Summary and future work

Aim of the study:

Inference on the jump part of quadratic variation.

Methodology:

- Difference of realised variance and realised multipower variation from tripower onwards

Future work:

- Multivariate extension (Work in progress with O. Barndorff-Nielsen and N. Shephard)