Modelling turbulent time series

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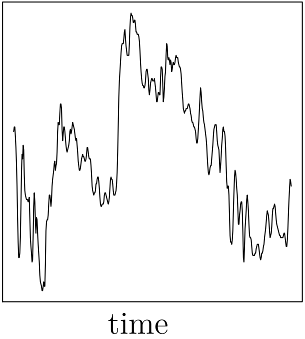
Outline

- Task
- Modelling the turbulent energy dissipation
- Case study: helium jet experiment
- Modelling the turbulent velocity
- Case study: helium jet experiment

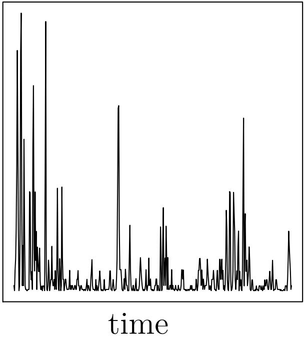
Task

Stationary time series

velocity



energy dissipation



Task

Stylized features:

- distribution of velocity increments $u_s = v_{t+s} - v_t$
- structure functions of velocity increments $S_n(s) = \mathrm{E}\{u_s^n\}$
- distribution of the energy dissipation ε_t
- energy dissipation correlators
- selfscaling of correlators
- statistics of the Kolmogorov variable

Task

Tools: Ambit Stochastics

- BSS-processes

$$v_t = \int_{-\infty}^{t} g\left(t-s\right)\sigma_s \mathrm{d}W_s + \beta \int_{-\infty}^{t} g\left(t-s\right)\sigma_s^2 \mathrm{d}s$$

- continuous cascades

$$\sigma_t^2 = \exp\left\{L(A_t)\right\}$$

Identification: semimartingale case

- BSS-processes

$$[v]_t = \int_0^t (\mathrm{d}v_s)^2 = g^2(0+) \int_0^t \sigma_s^2 \mathrm{d}s$$

- classical definition of the integrated energy dissipation

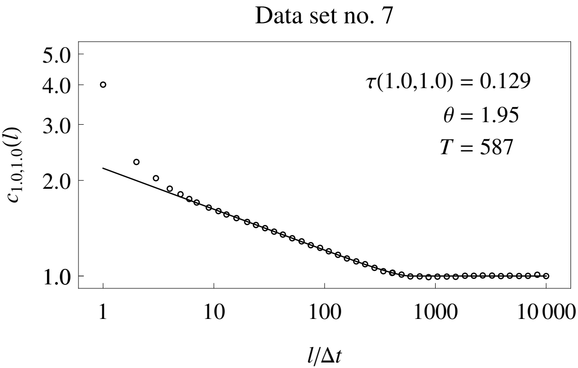
$$\int_0^t \varepsilon_s \mathrm{d}s \propto \int_0^t \left(\frac{\mathrm{d}v_s}{\mathrm{d}s} \right)^2 \mathrm{d}s$$

- stochastic analogue

$$\sigma^2 = \varepsilon$$

Correlators:

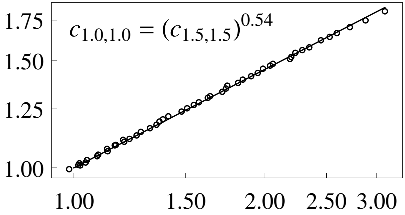
$$c_{n,m}(l) = \frac{\mathrm{E}\left\{\varepsilon_l^n \varepsilon_0^m\right\}}{\mathrm{E}\left\{\varepsilon_l^n\right\}\mathrm{E}\left\{\varepsilon_0^m\right\}}$$

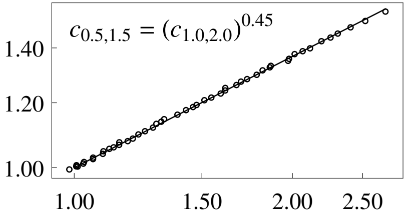


Modelling the turbulent energy dissipation

Self-scaling of correlators:

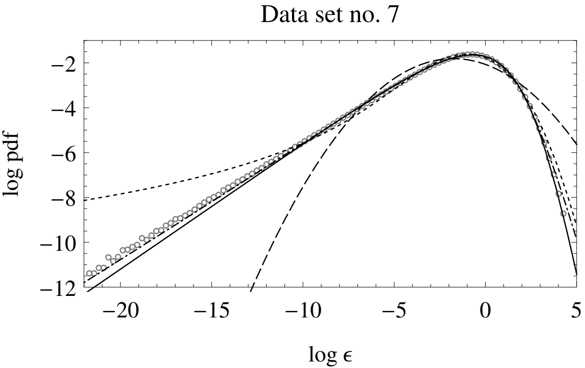
$$c_{n,m}(l) = c_{p,q}(l)^{\tau(p,q;n,m)}$$





Modelling the turbulent energy dissipation

Distribution of the logarithm of the energy dissipation:



Modelling the turbulent energy dissipation

Model:

homogeneous Lévy basis Z on $\mathbb{R}^2$, ambit set
$A(t) = A + (0, t)$

$$\varepsilon(t) = \exp\{Z(A(t))\}$$

where

$$A = \{(x, t) | 0 \leq t \leq T, |x| \leq q(t)\}$$

and

$$q(t) = \left(\frac{1 - (t/T)^{\Theta}}{1 + (t/(T/L))^{\Theta}} \right)^{1/\Theta}, \quad 0 \leq t \leq T$$

Modelling the turbulent energy dissipation

Properties:

- Moment generating function $K_X(s) = \log \mathrm{E}\{\exp(sX)\}$

$$K_{\log \varepsilon(t)}(s) = K_{Z'}(s)\mathrm{vol}(A)$$

- correlators

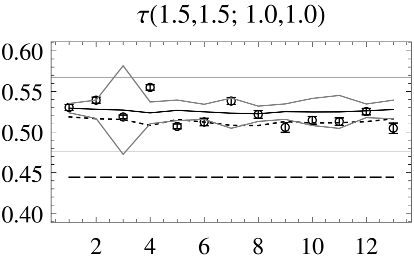
$$c_{p,q}(l) = \exp\left\{K(p,q)\mathrm{vol}\left(A(l) \cap A(0)\right)\right\}$$

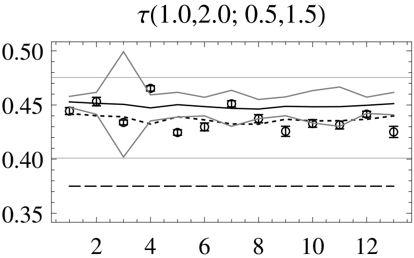
where

$$K(p,q) = K_{Z'}(p+q) - K_{Z'}(p) - K_{Z'}(q)$$

Case study: helium jet experiment

Prediction:





Modelling the turbulent velocity

Semimartingale model

- BSS-processes

$$v_t = \int_{-\infty}^{t} g\left(t-s\right)\sigma_s \mathrm{d}W_s + \beta \int_{-\infty}^{t} g\left(t-s\right)\sigma_s^2 \mathrm{d}s$$

- shifted 2-gamma kernel

$$g\left(x;x_0\right) = \left(h\left(\cdot;a_1,\nu_1,\lambda_1\right) * h\left(\cdot;a_2,\nu_2,\lambda_2\right)\right)\left(x+x_0\right)1_{\mathbb{R}_+}\left(x\right)$$

where

$$h\left(x;a,\nu,\lambda\right) = a\cdot x^{\nu-1}\exp\left(-\lambda x\right)1_{(0,\infty)}\left(x\right)$$

Modelling the turbulent velocity

Spectral density function:

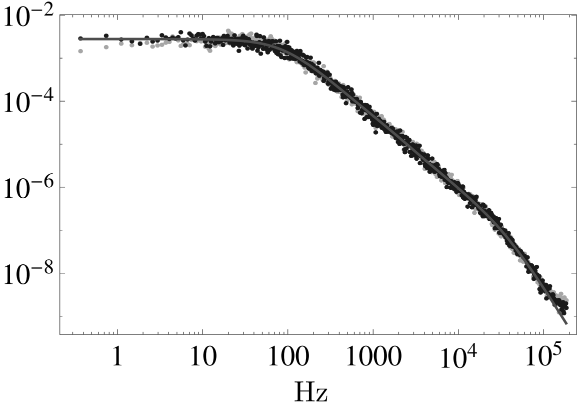
- for $x_0 = 0$

$$\widehat{r}_v(\omega) = a^2 \left(1 + \beta^2 \widehat{r}_{\sigma^2}(\omega)\right) \left(1 + \left(\frac{2\pi\omega}{\lambda_1}\right)^2\right)^{-\nu_1} \left(1 + \left(\frac{2\pi\omega}{\lambda_2}\right)^2\right)^{-\nu_2}$$

- for $x_0 = 0$ and $\beta = 0$

$$\widehat{r}_v(\omega) \propto \begin{cases} 1 & \omega \ll \lambda_1/2\pi \\ \omega^{-2\nu_1} & \lambda_1/2\pi \ll \omega \ll \lambda_2/2\pi \\ \omega^{-2(\nu_1+\nu_2)} & \omega \gg \lambda_2/2\pi. \end{cases}$$

Spectral density function:



Modelling the turbulent velocity

Skewness parameter β:

$$v_t = R_t + \beta Q_t$$

where

$$R_t = \int_{-\infty}^{t} g\left(t-s\right)\sigma_s \mathrm{d}W_s$$

and

$$Q_t = \int_{-\infty}^{t} g\left(t-s\right)\sigma_s^2 \mathrm{d}s$$

Modelling the turbulent velocity

Skewness parameter β:

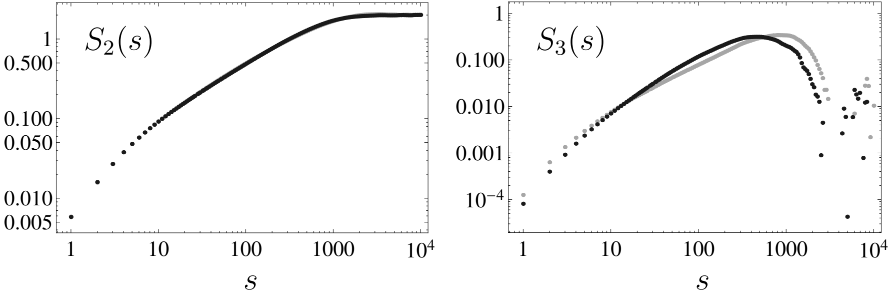
$$S_3(s) = \mathrm{E}\left\{(v_s - v_0)^3\right\} = 3\beta\mathrm{E}\left\{(\Delta_s R)^2 (\Delta_s Q)\right\} + \beta^3 \mathrm{E}\left\{(\Delta_s Q)^3\right\}$$

where

$$\Delta_s R = R_s - R_0, \;\; \Delta_s Q = Q_s - Q_0$$

Modelling the turbulent velocity

Structure functions:



Modelling the turbulent velocity

Iteration procedure:

- Start: simulation of σ, set $\beta = 0$, fit g from the sdf, simulate

$$R_t = \int_{-\infty}^{t} g\left(t - s\right) \sigma_s \mathrm{d}W_s, \quad Q_t = \int_{-\infty}^{t} g\left(t - s\right) \sigma_s^2 \mathrm{d}s$$

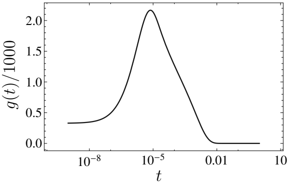
- Iteration: fit β from

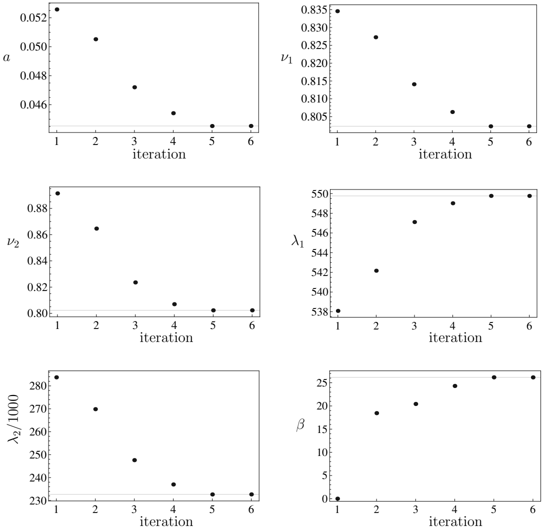
$$S_3(s) = 3\beta \mathrm{E}\left\{\left(\Delta_s R\right)^2 \left(\Delta_s Q\right)\right\} + \beta^3 \mathrm{E}\left\{\left(\Delta_s Q\right)^3\right\}$$

  and re-estimate g from the sdf
- redo until convergence

Case study: helium jet experiment

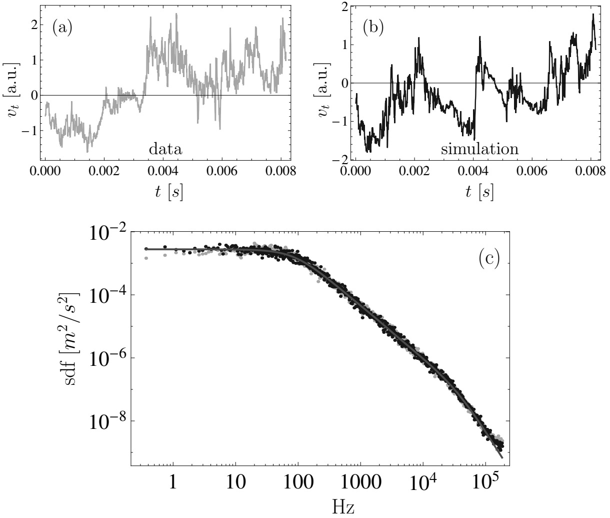
Parameters:





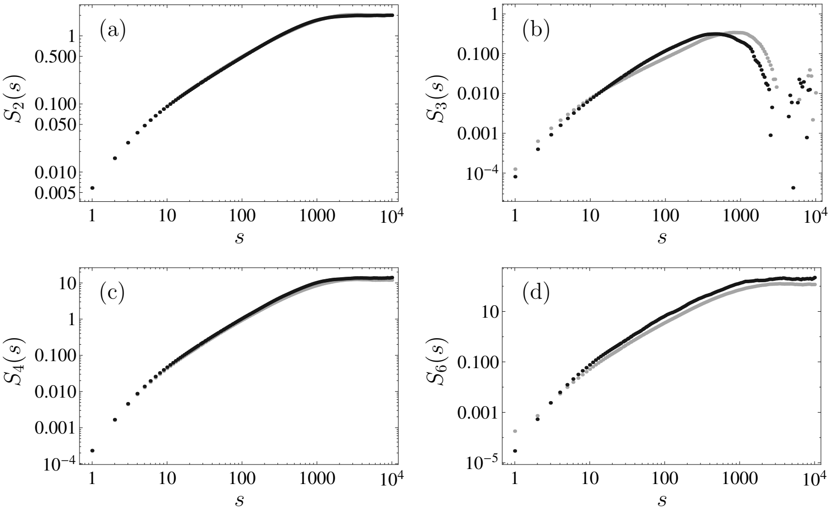
Case study: helium jet experiment

Spectral density function:



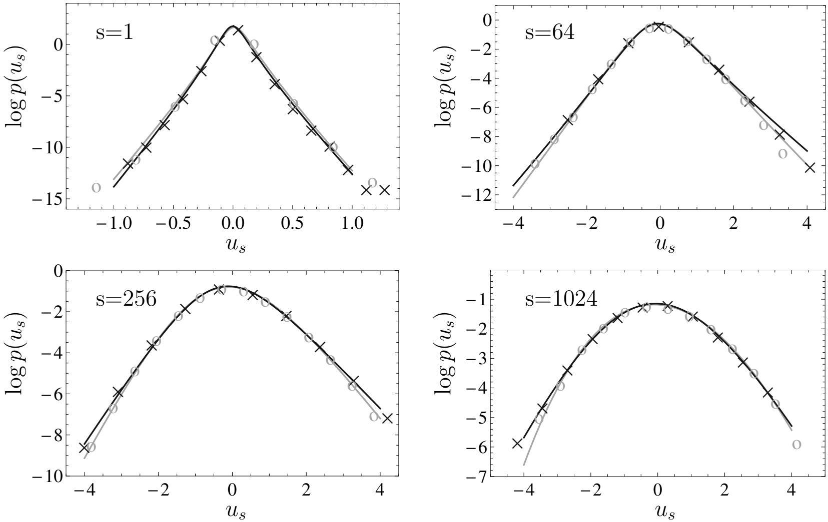
Case study: helium jet experiment

Structure functions:



Case study: helium jet experiment

Distribution of increments:

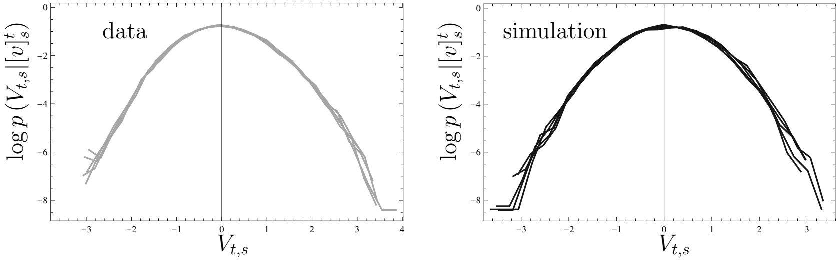


Case study: helium jet experiment

Kolmogorov variable:

$$V_{t,s} = \frac{v_{t+s/2} - v_{t-s/2}}{\left(\bar{v}\,[v]_s^t\right)^{1/3}}$$

where $[v]_s^t = [v]_{t+s/2} - [v]_{t-s/2}$



Case study: helium jet experiment

Correlators of the energy dissipation: