Functional convergence of random series and infinitely divisible processes

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Series representation of Lévy processes

Let $(X_t)_{t\in[0,1]}$ be a d-dim. Lévy process with series representation

$$X_t = \sum_{k=1}^{\infty} \Big(H(\Gamma_k, V_k)\mathbf{1}_{\{U_k \le t\}} - c_k t \Big) \qquad \text{a.s. } t \in [0,1]$$

where

1. (Γ_k) arrival times in Poisson process
2. (V_k) i.i.d. in $\mathcal{V}$
3. (U_k) i.i.d. $U(0,1)$-sequence
4. $H : [0,1] \times \mathcal{V} \to \mathbb{R}^d$ and $c_k \in \mathbb{R}^d$.

Applications:

1. Read structural properties of the process.
2. Simulations.

Robustness of the series representation

Let $(X_t)_{t\in[0,1]}$ be a d-dim. Lévy process with series representation

$$X_t = \sum_{k=1}^{\infty} \Big(H(\Gamma_k, V_k)\mathbf{1}_{\{U_k \le t\}} - c_k t \Big) \qquad \text{a.s. } t \in [0,1]. \qquad (1)$$

1. Kallenberg [1] and Rosiński [2] show that the series (1) converges uniformly a.s.
2. This implies

$$\Delta X_t = \sum_{k=1}^{\infty} H(\Gamma_k, V_k)\mathbf{1}_{\{U_k = t\}}$$

since $x \mapsto \Delta x$ is continuous in $\|\cdot\|_\infty$.

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[1] Kallenberg, O. (1974). Series of random processes without discontinuities of the second kind. *Ann. Probab.* 2.
[2] Rosiński, J. (2001). Series representations of Lévy processes from the perspective of point processes. In *Lévy Processes*.

Rosiński '11: *Is the series representation robust wrt. SDEs?*

$$X_t = \sum_{k=1}^{\infty} \Big(H(\Gamma_k, V_k) \mathbf{1}_{\{U_k \le t\}} - c_k t \Big),$$
$$X_t^n = \sum_{k=1}^{n} \Big(H(\Gamma_k, V_k) \mathbf{1}_{\{U_k \le t\}} - c_k t \Big).$$

1. $F \in C^2(\mathbb{R}^d; \mathbb{R}^d)$
2. $$dZ_t = F(Z_{t-})\, dX_t \qquad \text{and} \qquad dZ_t^n = F(Z_{t-}^n)\, dX_t^n.$$
3. Question $(\star)$: Will $Z_t^n \to Z_t$?

Question $(\star)$ does not follow from the above mentioned results, since the Itô map (solution map) $X \mapsto Z$ is discontinuous in $\|\cdot\|_\infty$.

However, the Itô map is continuous in the p-variation norm for $p < 2$!

Itô–Nisio theorem $\longrightarrow$ Functional convergence of random series

Definition (Itô–Nisio theorem)

1. For $i \in \mathbb{N}$ let $X_i = \{X_i(t) : t \in T\}$ be independent and symmetric stochastic processes with sample paths in a Banach space $(F, \|\cdot\|)$.
2. Suppose that there exists a stochastic process S with sample paths in F such that

$$\left\{\sum_{j=1}^{\infty} X_j(t)\right\}_{t\in T} \overset{d}{=} \{S(t)\}_{t\in T}.$$

Then, the series

$$\sum_{j=1}^{\infty} X_j \qquad \text{converge almost surely in } (F, \|\cdot\|).$$

Separable Banach spaces

1. For F separable, the Itô–Nisio theorem holds due to Itô and Nisio '68.
2. The original motivation for the Itô–Nisio theorem came showing uniform convergence of the Karhunen-Loève represention of the Brownian motion and other Gaussian processes. Use $F = C[0,1]$.
3. The proof relies heavily on the fact that probability measure on separable Banach spaces are convex tight.

Non-separable Banach spaces

1. The case of non-separable Banach spaces are especially important for stochastic processes with jumps.
2. If (X_t) is a Poisson process then the law of X is *not* concentrated on a separable subset of $D[0,1]$ or BV_p for all $p \geq 1$.
3. For F non-separable space the Itô–Nisio theorem holds sometimes holds and sometimes not.
4. It does *not hold* for the

$$\text{Hölder spaces } C^{0,\alpha}, \alpha \in (0,1] \quad \text{or} \quad \text{bounded sequences } \ell^\infty.$$

Theorem (B. and Rosiński [1])

The Itô–Nisio theorem holds for $(D[0,1], \|\cdot\|_\infty)$.

1. The theorem implies uniform converges of general càdlàg infinitely divisible processes (beyond Lévy processes).
2. $D[0,1]$ is separable under the Skorohod topology, but it does not help since probability measures on $D[0,1]$ are not convex tight due discontinuity of addition.

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[1] Basse-O'Connor, A. and J. Rosiński (2013). On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes. *Ann. Probab.* 41.

Bounded p-variation

Let BV_p be set of all càdlàg functions f of bounded p-variation

$$\|f\|_{BV_p} := \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p < \infty.$$

BV_p is a non-separable Banach space.

1. The Itô–Nisio theorem holds for BV_1, cf. [1].
2. For $1 < p < \infty$ the Itô–Nisio theorem does *not hold* for BV_p, cf. [2].

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[1] Jain, N, and D. Monrad (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete* 59.

[2] Jain, N, and D. Monrad (1983). Gaussian measures in B_p. *Ann. Probab.* 11.

The Wiener class

Let BV_p^* be the set of $f \in BV_p$ where

$$\lim_{\kappa} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \qquad \text{exists}$$

where $\kappa = \{0 = t_0 < \cdots < t_n = 1\}$ and the limit is in refinement of partitions.

1. Rough paths theory: A geometric rough path of order p is an element in the Wiener class $BV_p^*([0,1]; G)$.
2. $(BV_p^*, \|\cdot\|_{BV_p})$ is a non-separable Banach space
3. $BV_1 = BV_1^*$ and $BV_\infty = BV_\infty^*$
4. $$\bigcup_{\epsilon>0} BV_{p-\epsilon} \subsetneq BV_p^* \subsetneq BV_p, \qquad 1 < p < \infty.$$

Theorem

The Itô–Nisio theorem holds for the Wiener class BV_p^*.

An important ingredient in the proof is:

Lemma

The family of separable subsets of BV_p^* coincide for $\|\cdot\|_\infty$ and $\|\cdot\|_{BV_p}$.

1. The lemma is not true for BV_p.
2. Since $\overline{A}_{\|\cdot\|_\infty}$ is much larger than $\overline{A}_{\|\cdot\|_{BV_p}}$, the result is somehow surprising.

Theorem

Let $X = (X_t)$ be an infinitely divisible process.
Let H be a representation of the Lévy measure of X, which has series representation

$$X_t = \sum_{k=1}^{\infty} \Big(H(t, \Gamma_k, V_k) - c_k(t) \Big). \tag{2}$$

Suppose that $X \in BV_p^*$ a.s. Then the series (2) converges in p-variation norm a.s.

1. Note that the assumption $X \in BV_p^*$ is always satisfied if $X \in BV_q$ for some $q < p$.
2. Conditionally on (Γ_k), the summans are independent.
3. In view of the Itô–Nisio theorem on BV_p^*, the difficulty consist in dealing with the non-symmetry of the summans.

Proposition

Let (X_t) be a Lévy process and $p < 2$.
Then $X \in BV_p$ a.s. if and only if $X \in BV_p^*$.

Theorem

Let (X_t) be a d-dim. Lévy process of bounded p-variation for $p < 2$. Let $F \in C^2(\mathbb{R}^d, \mathbb{R}^d)$,

$$dZ_t = F(Z_{t-})\, dX_t \qquad \text{and} \qquad dZ^n = F(Z^n_{t-})\, dX^n_t.$$

Then

$$Z^n \to Z \qquad \text{in p-variation norm a.s.}$$

Proof: Proposition $\Rightarrow X \in BV_p^*$ a.s. $\Rightarrow (X_t^n) \to (X_t)$ in p-variation norm by the functional converges for series representation of ID process in BV_p^*. The continuity of the Itô map in $\| \cdot \|_p$ concludes the proof. □

General conditions of the Itô–Nisio theorem

Theorem

The Itô–Nisio theorem holds for F if at least one of the following two conditions (i) or (ii) are satisfied:

(i) $B_{F^*}(0,1)$ *is sequentially weak* compact*

(ii) *No subspace of F is isomorphic to c_0.*

Conversely, if the Itô–Nisio theorem holds for every subspace of F, then no subspace of F is isomorphic to ℓ^∞.

1. (i) is satisfied for all separable Banach space due to the Banach–Alaoglu theorem.
2. (ii) is satisfied for some separable and some non-separable Banach space.

More examples

Theorem

For every $d \geq 1$ then Itô–Nisio theorem holds for $D([0,1]^d;\mathbb{R})$.

Theorem

Let U and V be separable Banach spaces.

Then Itô–Nisio theorem holds for $\mathcal{L}(U,V)$ if and only if no subspace of $\mathcal{L}(U,V)$ are isomorphic to ℓ^∞.

Itô–Nisio theorem and Lévy measures

Theorem

Let $(X_t)_{t\in T}$ be an infinitely divisible process with Lévy measure ν. Moreover, let F be a Banach space such that $X \in F$ a.s.

If the Itô–Nisio theorem holds for F then $\nu(B_F(0,r)) < \infty$ for every $r > 0$.

Hence, the unit ball can always be used when the Itô–Nisio theorem holds.

1. Without the Itô–Nisio theorem we only have existence for a $r > 0$ such that $\nu(B_F(0,r)) < \infty$, cf. [1].
2. For $F = \ell^\infty$, the conclusion of the theorem does not hold.

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[1] Rosiński, J. and G. Samorodnitsky (1993). Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Ann. Probab.* 21.

Thank you for your attention!