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CARMA processes in Hilbert space

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Overview

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Overview

- Introduce the multivariate Ornstein-Uhlenbeck process $\{\mathbf{Z}(t)\}_{t\geq 0}$ with values in $\mathbb{R}^p$ for $p \in \mathbb{N}$ by

$$d\mathbf{Z}(t) = C_p\mathbf{Z}(t)dt + \mathbf{e}_p dL(t), \; \mathbf{Z}(0) = \mathbf{Z}_0 \in \mathbb{R}^p.$$

- L is real-valued, square integrable Lévy process with zero mean
- $\{\mathbf{e}_i\}_{i=1}^p$ is the canonical basis in $\mathbb{R}^p$, while the $p \times p$ matrix C_p takes the particular form

$$C_p = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 1 \\ -\alpha_p & -\alpha_{p-1} & . & . & . & . & -\alpha_1 \end{bmatrix},$$

for constants $\alpha_i > 0$, $i = 1, \ldots, p$.

- Define a continuous-time autoregressive process of order p (CAR(p)-process) by

$$X(t) = \mathbf{e}_1^\top \mathbf{Z}(t), \; t \geq 0,$$

- For $q \in \mathbb{N}$ with $p > q$, we define a CARMA(p, q)-process by

$$X(t) = \mathbf{b}^\top \mathbf{Z}(t), \; t \geq 0,$$

- Here, $\mathbf{b} = (b_0, b_1, ..., b_{q-1}, 1, 0, .., 0)^\top \in \mathbb{R}^p$
- Multivariate extensions of CARMA processes: Stelzer et al.

Overview

Motivating example: the wave equation

$$\frac{\partial^2 Y(t,x)}{\partial t^2} = \frac{\partial^2 Y(t,x)}{\partial x^2} + \frac{\partial L(t,x)}{\partial t}, \qquad t>0, x\in(0,1)$$

- 2nd order (in time) PDE: with $\Delta = \partial^2/\partial x^2$

$$d\begin{bmatrix} Y(t,x) \\ \frac{\partial Y(t,x)}{\partial t}\end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Id} \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} Y(t,x) \\ \frac{\partial Y(t,x)}{\partial t}\end{bmatrix} dt + \begin{bmatrix} 0 \\ dL(t,x)\end{bmatrix}$$

- OU-dynamics in Hilbert space: $H := H_1 \times H_2$
  - $H_2 := L^2(0,1)$, basis $\{e_n\}_{n\in\mathbb{N}}$ with $e_n(x) := \sqrt{2}\sin(\pi n x)$
  - $L(t,\cdot)$ is an H_2-valued Lévy process
  - $H_1 \subset L^2(0,1)$, where $|f|_1^2 := \pi^2 \sum_{n=1}^{\infty} n^2 \langle f, e_n\rangle_2^2 < \infty$

General definition

- $H := H_1 \times H_2 \times \ldots \times H_p$, $p \in \mathbb{N}$
  - H_i's are separable Hilbert spaces
- The projection operator $\mathcal{P}_i : H \to H_i$: $\mathcal{P}_i \mathbf{x} = x_i$ for $\mathbf{x} \in H$, $i = 1, \ldots, p$
  - Adjoint $\mathcal{P}_i^* : H_i \to H$: $\mathcal{P}_i^* x = (0, \ldots, 0, x, 0, \ldots, 0)$ for $x \in H_i$, where the x appears in the ith coordinate
- $L(t)$ H_p-valued Lévy process
  - Square integrable with zero mean
  - Covariance operator $Q \in L(H_p)$
- H-valued OU process

$$d\mathbf{Z}(t) = \mathcal{C}_p \mathbf{Z}(t) dt + \mathcal{P}_p^* dL(t), \ \mathbf{Z}(0) := \mathbf{Z}_0 \in H.$$

- $\mathcal{C}_p : H \to H$ linear operator (unbounded), represented as a $p \times p$ matrix of operators

$$\mathcal{C}_p = \begin{bmatrix} 0 & I_p & 0 & . & . & . & 0 \\ 0 & 0 & I_{p-1} & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & I_2 \\ A_p & A_{p-1} & . & . & . & . & A_1 \end{bmatrix}.$$

- $A_i : H_{p+1-i} \to H_p, i = 1, \ldots, p$ are p (unbounded) densely defined linear operators, and $I_i : H_{p+2-i} \to H_{p+1-i}, i = 2, \ldots, p$ are another $p - 1$ (unbounded) densely defined linear operators.

- Assume $\mathcal{C}_p$ is densely defined operator
  - If $H_1 = \ldots = H_p$ and $I_i = \text{Id}$, $\text{Dom}(\mathcal{C}_p)$ is dense

$$Dom(\mathcal{C}_p) = Dom(A_p) \times Dom(A_{p-1}) \times \ldots \times Dom(A_1)$$

- From theory of SPDEs (see Peszat and Zabczyk):

Proposition

Assume that $\mathcal{C}_p$ is the generator of a C_0-semigroup $\{\mathcal{S}_p(t)\}_{t\geq 0}$ on H. Then the H-valued stochastic process $\mathbf{Z}$ is given by

$$\mathbf{Z}(t) = \mathcal{S}_p(t)\mathbf{Z}_0 + \int_0^t \mathcal{S}_p(t-s)\mathcal{P}_p^* \, dL(s)$$

Definitions

- General CARMA

Definition

Let U be a separable Hilbert space. For $\mathcal{L}_U \in L(H, U)$, define the U-valued stochastic process $X(t)$ by

$$X(t) := \mathcal{L}_U \mathbf{Z}(t), t \geq 0$$

We call $X(t)$ a CARMA$(p, U, \mathcal{L}_U)$-process.

- A CARMA$(p, H_1, \mathcal{P}_1)$-process $X(t)$ is called an H_1-valued CAR(p)-process.

$$X(t) = \mathcal{P}_1 \mathbf{Z}(t) = Z_1(t)$$

Overview

CAR(p) as pth order differential equation in H

- By definition of the operator matrix $\mathcal{C}_p$

$$Z_1'(t) = I_p Z_2(t), Z_2'(t) = I_{p-1} Z_3(t), \ldots, Z_{p-1}'(t) = I_2 Z_p(t)$$

and

$$Z_p'(t) = A_p Z_1(t) + \cdots A_1 Z_1(t) + L'(t).$$

- Assume there exist $p-1$ linear (unbounded) operators $B_1, B_2, \ldots, B_{p-1} : H_1 \to H_1$,

$$I_p \cdots I_2 A_q = B_q I_p I_{p-1} \cdots I_{q+1}, \quad q = 1, \ldots, p-1 \tag{1}$$

Additionally, define the operator $B_p : H_1 \to H_1$ as

$$B_p := I_p \cdots I_2 A_p. \tag{2}$$

- By iteration,

$$Z_1^{(q)}(t) = I_p I_{p-1} \cdots I_{p-(q-1)} Z_{q+1}(t), \quad q = 1, \ldots, p-1$$

- Thus,

$$\begin{aligned} Z_1^{(p)}(t) &= \frac{d}{dt} Z_1^{(p-1)}(t) = I_p \cdots I_2 Z_p'(t) \\ &= I_p \cdots I_2 A_p Z_1(t) + I_p \cdots I_2 A_{p-1} Z_2(t) + \cdots + I_p \cdots I_2 A_1 Z_p(t) \\ &\quad + I_p \cdots I_2 L'(t) \\ &= B_p Z_1(t) + B_{p-1} Z_1'(t) + B_{p-2} Z_1^{(2)}(t) + \ldots + B_1 Z_1^{(p-1)}(t) \\ &\quad + I_p \cdots I_2 L'(t). \end{aligned}$$

- Introduce the operator-valued pth-order polynomial $Q_p(\lambda)$ for $\lambda \in \mathbb{C}$,

$$Q_p(\lambda) = \lambda^p - B_1\lambda^{p-1} - B_2\lambda^{p-2} - \cdots - B_{p-1}\lambda - B_p.$$

- In conclusion, a CAR(p) process $X(t) = Z_1(t)$ can be viewed as the solution of the pth-order differential equation,

$$Q_p\left(\frac{d}{dt}\right)X(t) = I_p \cdots I_2 L'(t).$$

- If $H_1 = ... = H_p$, $\mathcal{C}_p$ is a bounded operator and $I_i = \text{Id}$ for $i = 2, \ldots, p$, we have

$$A_q = B_q \,, \quad q = 1, \ldots, p$$

- Further suppose X is a CARMA(p,$H_1, \mathcal{L}_{H_1}$)
  - I.e., $H = H_1^{\times p}$ and $U = H_1$
  - $\mathcal{L}_{H_1}$ is a vector-valued operator $\mathcal{L}_{H_1} := (M_1, \ldots, M_p)$, where $M_i \in L(H_1), i = 1, ..., p$.
- Assume M_i commutes with A_j for all i, j

$$X(t) = \sum_{i=1}^{p} M_i Z_i(t)$$

- Using the relationships for $Z_1, ..., Z_p$ and the commutation assumptions.......

$$Q_p\left(\frac{d}{dt}\right)X(t) = R_{p-1}\left(\frac{d}{dt}\right)L'(t)$$

- Operator-valued $(p-1)$th-order polynomial $R_{p-1}(\lambda), \lambda \in \mathbb{C}$,

$$R_{p-1}(\lambda) = M_p\lambda^{p-1} + M_{p-1}\lambda^{p-2} + \cdots + M_2\lambda + M_1 .$$

- Hence, informally, a CARMA$(p, H_1, \mathcal{L}_{H_1})$-process $\{X(t)\}_{t\geq 0}$ can be represented by an autoregressive polynomial operator Q_p and a moving average polynomial operator R_{p-1}.
  - With rather strong conditions on commutativity on the A's and M's...

Functional AR(p) process

- Focus on X being CAR(p) process, i.e. $\mathcal{L}_{H_1} = \mathcal{P}_1$
- For $\delta > 0$, $t_i = i \cdot \delta, i = 0, 1, 2.....$
- Introduce nth-order forward differencing operator Δ_δ^n

$$\Delta_\delta^n f(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(t + (n-k)\delta)$$

  for a function f and $n \in \mathbb{N}$.
- Define (formally) a time series $\{x_i\}_{i=0}^{\infty}$ in H_1 by

$$Q_p\left(\frac{\Delta_\delta}{\delta}\right) x_i = \epsilon_i, \qquad \epsilon_i := \frac{1}{\delta}(L(t_{i+1}) - L(t_i)).$$

  - We use the notation $x_i = x(t_i)$ when applying Δ_δ
  - Initial values $x_0, \ldots, x_{p-1} \in H$ given

Proposition

$\{x_i\}_{i=0}^{\infty}$ is an AR(p) process in H_1 with dynamics

$$x_{i+p} = \sum_{q=1}^{p} \widetilde{A}_q x_{i+(p-q)} + \delta^p \epsilon_i$$

where

$$\widetilde{A}_q = (-1)^{q+1} \binom{p}{q} Id + \sum_{k=1}^{q} \delta^k A_k (-1)^{q-k} \binom{p-k}{q-k}, q = 1, \ldots, p$$

- Result can be extenbded to unbounded case!

Pathwise regularity

Proposition

For $p \in \mathbb{N}$ with $p > 1$, assume that $\mathcal{C}_p$ is the generator of a C_0-semigroup $\{\mathcal{S}_p(t)\}_{t\geq 0}$. Then the H_1-valued CAR(p) process X has the representation

$$X(t) = \mathcal{P}_1\mathcal{S}_p(t)\mathbf{Z}_0 + \mathcal{P}_1\mathcal{C}_p \int_0^t \int_0^u \mathcal{S}_p(u-s)\mathcal{P}_p^* \, dL(s) \, du,$$

for all $t \geq 0$.

Proof.

Representation of semigroup and generator:

$$\mathcal{S}_p(t) = \mathrm{Id} + \mathcal{C}_p \int_0^t \mathcal{S}_p(s) ds.$$

Thus,

$$X(t) = \mathcal{P}_1 \mathcal{S}_p(t) \mathbf{Z}_0 + \mathcal{P}_1 \int_0^t \mathcal{C}_p \int_s^t \mathcal{S}_p(u-s) \mathcal{P}_p^* \, du \, dL(s).$$

Show that $\mathcal{C}_p$ can be pulled out of $dL(s)$-integral. Invoke stochastic Fubini theorem. ∎

Concluding remarks

- When is $\mathcal{C}_p$ generating a C_0-semigroup?
  - Special case: $A_1, \dots, A_p$, $I_2, \dots, I_p$ are bounded operators
  - Partial extension: A_1 unbounded....recursive representation of semigroup
- Existence of limiting distribution for X?
  - Semigroup $\mathcal{S}_p$ must be exponentially stable
  - If $\mathcal{C}_p$ is bounded, $\mathcal{S}_p(t)$ exponentially stable iff $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(\mathcal{C}_p)$, the spectrum of $\mathcal{C}_p$
- Characteristic functional (cumulant) of the limiting distribution is available

Thank you for your attention!

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