Intermittency for the stochastic heat equation with Lévy noise

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Joint work with Péter Kevei (University of Szeged)

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The classical heat equation

$$\begin{aligned}\partial_t Y(t,x) &= \frac{\kappa}{2}\Delta Y(t,x) + F(t,x), \quad t>0,\ x\in\mathbb{R}^d,\\ Y(0,x) &= f(x).\end{aligned}$$

- F smooth and compactly supported force function
- f bounded measurable initial condition
- κ positive diffusion constant

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Solution:

$$Y(t,x) = \int_{\mathbb{R}^d} g(t,x-y)f(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} g(t-s,x-y)F(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

where

$$g(t,x) = \frac{\mathrm{e}^{-\frac{|x|^2}{2\kappa t}}}{(2\pi\kappa t)^{\frac{d}{2}}}\mathbf{1}_{\{t>0\}}$$

Stochastic heat equation with multiplicative Lévy noise

$$\boxed{\begin{aligned}\partial_t Y(t,x) &= \frac{\kappa}{2}\Delta Y(t,x) + \sigma(Y(t,x))\dot{L}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d,\\ Y(0,x) &= f(x).\end{aligned}}$$

- σ globally Lipschitz function
- $\dot{L}$ **Lévy** space–time white noise (= distributional derivative of a Lévy sheet L)

$$L(\mathrm{d}t,\mathrm{d}x) = \int_{\mathbb{R}} z\,(\mu - \nu)(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) \qquad \textbf{Lévy basis}$$

μ Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ with intensity measure

$$\nu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z) = \mathrm{d}t\,\mathrm{d}x\,\lambda(\mathrm{d}z), \quad \lambda(\{0\}) = 0, \quad \int_{\mathbb{R}} |z| \wedge |z|^2\,\lambda(\mathrm{d}z) < \infty$$

Mild formulation:

$$
\boxed{
\begin{aligned}
Y(t,x) &= Y_0(t,x) + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\sigma(Y(s,y))\, L(\mathrm{d}s, \mathrm{d}y),\\
Y_0(t,x) &= \int_{\mathbb{R}^d} g(t, x-y) f(y)\, \mathrm{d}y.
\end{aligned}
}
\tag{SHE-L}
$$

where

$$
g(t,x) = \frac{1}{(2\pi\kappa t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{2\kappa t}} \mathbf{1}_{\{t>0\}}
$$

Existence and uniqueness

Theorem (Saint Loubert Bié 98)

If L has no Gaussian part, and there exists $p \in [1, 1+\frac{2}{d})$ with

$$\int_{\mathbb{R}} |z|^p \, \lambda(\mathrm{d}z) < \infty,$$

then (SHE) has a unique solution satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p] < \infty \quad \forall T \geq 0.$$

Remarks:

1. With **Gaussian noise**, (SHE) only has a solution in $d = 1$ (Walsh 86).
2. Extensions to heavy-tailed (e.g. **stable**) noises possible (Chong 17).

Problem formulation

What is the behavior of $\mathbb{E}[|Y(t,x)|^p]$ as $t \to \infty$?

Definition:

1. **Upper and lower moment Lyapunov exponents:**

$$\overline{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}^d} \log \mathbb{E}[|Y(t,x)|^p],$$

$$\underline{\gamma}(p) := \liminf_{t\to\infty} \frac{1}{t} \inf_{x\in\mathbb{R}^d} \log \mathbb{E}[|Y(t,x)|^p].$$

2. The solution Y is **weakly intermittent of order** $\boldsymbol{p}$ if

$$0 < \underline{\gamma}(p) \leq \overline{\gamma}(p) < \infty.$$

Why intermittency?

From (Bertini & Cancrini 95):

Let $Y(t,x)$ be **nonnegative** and **stationary and ergodic** in x such that

$$\mathbb{E}[Y(t,x)] = \text{constant}$$

$$\gamma(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[Y(t,x)^p] \quad \text{exists for all } p \geq 1$$

$$\gamma(p) \to \infty, \quad p \nearrow p_{\max} := \sup\{p \geq 1 \colon \gamma(p) < \infty\}$$

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- In particular, $\gamma(1) = 0$.
- Weak intermittency of order $p > 1$ implies that

$$p \mapsto \frac{\gamma(p)}{p}$$

is strictly increasing on the set $\{p \geq 1 \colon \gamma(p) < \infty\}$.

Why intermittency?

- Assume $\gamma(p) > 0$, take $\alpha \in (0, \gamma(p)/p)$ and define

$$B_{t,\alpha} = \{x\colon\, Y(t,x) > \mathrm{e}^{\alpha t}\}, \quad \rho_{t,\alpha} = \lim_{R\to\infty} \frac{\mathrm{Leb}(B_{t,\alpha} \cap \{|x| \le R\})}{\mathrm{Leb}(\{|x| \le R\})}$$

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- **Ergodic theorem**:

$$\begin{aligned} \rho_{t,\alpha} &= \mathbb{P}[Y(t,x) > \mathrm{e}^{\alpha t}] \\ &\leq \mathrm{e}^{-\alpha t}\mathbb{E}[Y(t,x)] \\ &= C\mathrm{e}^{-\alpha t}. \end{aligned}$$

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- Write

$$\mathbb{E}[Y(t,x)^p] = \mathbb{E}[Y(t,x)^p \mathbf{1}_{B_{t,\alpha}}(x)] + \mathbb{E}[Y(t,x)^p \mathbf{1}_{B_{t,\alpha}^{\mathrm{C}}}(x)]$$

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- Hence:

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- If $p_n \nearrow p_{\max}$ and

$$\frac{\gamma(p_n)}{p_n} < \alpha_n < \frac{\gamma(p_{n+1})}{p_{n+1}} \qquad (\alpha_n \nearrow \infty),$$

there exist

$$B_{t,\alpha_1} \supset B_{t,\alpha_2} \supset B_{t,\alpha_3} \supset \ldots, \qquad \rho_{t,\alpha_n} \leq e^{-\alpha_n t},$$

with

$$e^{\gamma(p_n)t} \sim \mathbb{E}[Y(t,x)^{p_n}] \sim \mathbb{E}[Y(t,x)^{p_n} \mathbf{1}_{B_{t,\alpha_n}}]$$

Review: The Gaussian case

Theorem

Assume $d = 1$ and

$$0 < \inf_{x\in\mathbb{R}} f(x) \le \sup_{x\in\mathbb{R}} f(x) < \infty, \qquad \inf_{x\in\mathbb{R}\setminus\{0\}} |\sigma(x)/x| > 0.$$

Then:

1. (Foondun & Khoshnevisan 09): For every $p \ge 2$, we have
$$0 < \underline{\gamma}(p) \le \overline{\gamma}(p) < \infty.$$
2. (Bertini & Cancrini 95): If $f \equiv c > 0$ and $\sigma(x) = x$, then
$$\underline{\gamma}(p) \ge \frac{p(p^2-1)}{24\kappa}, \quad p \in \mathbb{N}.$$

Many extensions: rough initial data (Chen & Dalang 15), colored noise (Foondun & Khoshnevisan 11), ...

Lévy case: No high moments

Theorem (C. & Kevei 17)

Assume that $\lambda \not\equiv 0$ and there exists $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^d$ with

$$\sigma(Y_0(t_0, x_0)) \neq 0.$$

Then

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p] = +\infty$$

for all $T > t_0$ and $p \geq 1 + 2/d$.

In particular:

- No high moments even if jumps have nice moments.
- In dimension $d = 1$, only moments up to order $< \mathbf{3}$.
- In dimensions $d \geq 2$, the second moment is **infinite**.

Challenges:

In order to investigate the weak intermittency with Lévy noise, one has to consider in $d \geq 2$

1. **non-integer** moments,
2. moment orders **less than 2**, in fact, between $\mathbf{(1, 2)}$.

Results

Theorem (C. & Kevei 17)

Let Y be the solution to (SHE), and $\int_{\mathbb{R}} |z|^{1+2/d} \lambda(\mathrm{d}z) < \infty$,

$$0 < \inf_{x\in\mathbb{R}} f(x) \le \sup_{x\in\mathbb{R}} f(x) < \infty, \qquad \inf_{x\in\mathbb{R}\setminus\{0\}} |\sigma(x)/x| > 0.$$

1. For every $p \in [1, 1+2/d)$, we have

$$\overline{\gamma}(p) = \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}^d} \log \mathbb{E}[|Y(t,x)|^p] < \infty.$$

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But: In dimension $d = 1$ we can take $p_0 = 1$.

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Open: $p_0 = 1$ in higher dimensions?

Sketch of proof for the lower bound

Key lemma: Lower moment bound for Poisson integrals

Consider a Poisson random measure μ on $\mathbb{R}_+ \times E$ with intensity measure ν. Then for every $p \in (1,2)$ there is a universal constant $C_p > 0$ such that for any predictable process $W = W(\omega, t, x)$,

$$\mathbb{E}\left[\left|\iint_{\mathbb{R}_+ \times E} W(t,x)\,(\mu - \nu)(\mathrm{d}t, \mathrm{d}x)\right|^p\right] \geq C_p \frac{\iint_{\mathbb{R}_+ \times E} \mathbb{E}[|W(t,x)|^p]\,\nu(\mathrm{d}t, \mathrm{d}x)}{(1 \vee \nu(\mathbb{R}_+ \times E))^{1-p/2}}$$

Sketch of proof for the lower bound

Proof of Theorem for $d \geq 2$:

Step 1: For fixed (t, x) consider the truncation $\mu_1 = \mu_1^{(t,x)}$ of μ:

$$\mu_1(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \mathbf{1}_{[0,t]}(s)\mathbf{1}_{\{g(t-s,x-y)>1\}}\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(z)\,\mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$$

Then, by the BDG-inequalities,

$$\begin{aligned}
\mathbb{E}[|Y(t,x)|^p] &\gtrsim 1 + \mathbb{E}\left[\left(\iiint_0^t g^2(t-s,x-y)Y^2(s,y)z^2\,\mu(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{p/2}\right] \\
&\gtrsim 1 + \mathbb{E}\left[\left(\iiint_0^t g^2(t-s,x-y)Y^2(s,y)z^2\,\mu_1(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right)^{p/2}\right] \\
&\gtrsim 1 + \mathbb{E}\left[\left|\iiint_0^t g(t-s,x-y)Y(s,y)z\,(\mu_1-\nu_1)(\mathrm{d}s,\mathrm{d}y,\mathrm{d}z)\right|^p\right] \\
\text{(Lemma)} &\gtrsim 1 + \iint_0^t \frac{g^p(t-s,x-y)\mathbf{1}_{\{g(t-s,x-y)>1\}}}{\left(\iint_0^\infty \mathbf{1}_{\{g(s,y)>1\}}\,\mathrm{d}s\,\mathrm{d}y\right)^{1-p/2}}\mathbb{E}[|Y(s,y)|^p]\,\mathrm{d}s\,\mathrm{d}y
\end{aligned}$$

Sketch of proof for the lower bound

Step 2: In particular, for

$$I_p(t) := \inf_{x \in \mathbb{R}^d} \mathbb{E}[|Y(t,x)|^p],$$

we have

$$I_p(t) \gtrsim 1 + \int_0^t w_p(t-s) I_p(s)\, \mathrm{d}s$$

where

$$w_p(t) = \frac{\int_{\mathbb{R}^d} g^p(t,x) \mathbf{1}_{\{g(t,x)>1\}}\, \mathrm{d}x}{\left(\iint_0^\infty \mathbf{1}_{\{g(s,y)>1\}}\, \mathrm{d}s\, \mathrm{d}y\right)^{1-p/2}}.$$

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Renewal Theory: $I_p(t)$ grows exponentially fast in t if

$$\int_0^\infty w_p(t)\,\mathrm{d}t \quad \text{is large enough.}$$

Sketch of proof for the lower bound

Step 3: In order to achieve this, observe that

$$\int_0^\infty w_p(t)\,\mathrm{d}t = \frac{\iint_0^\infty g^p(t,x)\mathbf{1}_{\{g(t,x)>1\}}\,\mathrm{d}t\,\mathrm{d}x}{(\iint_0^\infty \mathbf{1}_{\{g(s,y)>1\}}\,\mathrm{d}s\,\mathrm{d}y)^{1-p/2}} \longrightarrow \infty$$

as $p \to 1 + 2/d$.

Sketch of proof for the lower bound

Proof of Theorem for $d = 1$: We fix $p \in (1,3)$ and consider

$$\mu_\epsilon(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z) = \mathbf{1}_{[0,t]}(s)\mathbf{1}_{\{g(t-s,x-y)>\epsilon\}}\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(z)\,\mu(\mathrm{d}s, \mathrm{d}y, \mathrm{d}z)$$

for small ϵ.

Indeed:

$$\int_0^\infty w_\epsilon(t)\,\mathrm{d}t = \frac{\iint_0^\infty g^p(t,x)\mathbf{1}_{\{g(t,x)>\epsilon\}}\,\mathrm{d}t\,\mathrm{d}x}{(\iint_0^\infty \mathbf{1}_{\{g(s,y)>\epsilon\}}\,\mathrm{d}s\,\mathrm{d}y)^{1-p/2}} \overset{\text{Calculation}}{=} \epsilon^{-p/2} \longrightarrow \infty$$

as $\epsilon \to 0$.

Asymptotics of Lyapunov exponents

Theorem: Gaussian noise

Assume $d = 1$ and

$$0 < \inf_{x\in\mathbb{R}} |f(x)| \leq \sup_{x\in\mathbb{R}} |f(x)| < \infty, \qquad \inf_{x\in\mathbb{R}\setminus\{0\}} |\sigma(x)/x| > 0.$$

Then:

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Asymptotics of Lyapunov exponents

Theorem (C. & Kevei 17)

Under the previous assumptions,

1.
$$0 < \liminf_{n\to\infty} \frac{1}{n \log n} \log \underline{\gamma}(1 + \tfrac{2}{d} - \tfrac{1}{n}) \leq \limsup_{n\to\infty} \frac{1}{n\log n} \log \overline{\gamma}(1 + \tfrac{2}{d} - \tfrac{1}{n}) < \infty$$

2.
$$0 < \liminf_{\kappa\to 0} \kappa^{\frac{p-1}{1+2/d-p}} \underline{\gamma}(p) \leq \limsup_{\kappa\to 0} \kappa^{\frac{p-1}{1+2/d-p}} \overline{\gamma}(p) < \infty$$

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Lévy: $\underline{\gamma}(1 + \frac{2}{d} - \frac{1}{n}) \approx n^n$ $\qquad$ $\underline{\gamma}(p;\kappa) \approx \kappa^{-\frac{p-1}{1+2/d-p}}$

Gauss: $\underline{\gamma}(n) \approx n(n^2 - 1)$ $\qquad$ $\underline{\gamma}(p;\kappa) \approx \kappa^{-1}$

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Conclusion

Intermittency/chaotic behavior **much stronger** in the Lévy case!

Thank you very much!

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