General limit theory for Lévy driven moving average processes

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joint work with A. Basse-O'Connor and M. Podolskij

Conference on Ambit Fields and Related Topics

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AARHUS UNIVERSITY

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Section 1

Introduction

Ambit fields as introduced by Barndorff-Nielsen and Schmiegel (2007):

$$X_t(x) = \int_{A_t(x)} g(t,s,x,\xi)\sigma_s(\xi)L(\mathrm{d}s,\mathrm{d}\xi) + \int_{D_t(x)} q(t,s,x,\xi)a_s(\xi)\,\mathrm{d}s\,\mathrm{d}\xi,$$

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In this talk we consider **Lévy driven moving average** (LDMA) processes of the form

$$X_t = \int_{-\infty}^{t} g(t-s) - g_0(-s)dL_s,$$

where g and g_0 are deterministic functions, and L is a Lévy process on the real line.

Motivation beyond ambit fields: Linear fractional stable motion

$$X_t = \int_{-\infty}^{t} (t-s)^{\alpha} - (-s)_+^{\alpha} dL_s,$$

where the driving Lévy process is symmetric β-stable, and $x_+ := x 1_{x>0}$. and $\alpha \in (-1/\beta, 1-1/\beta) \setminus \{0\}$.

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- Stationary, self-similar process of order $H := \alpha + 1/\beta$ with β-stable marginal distribution.
- For $\beta = 2$, X is fractional Brownian motion with Hurst parameter $H = \alpha + 1/2$.
- X is Lévy driven moving average process with $g(x) = g_0(x) =: x_+^{\alpha}$.
- Limit theory presented in this talk is applicable when $\beta > 1$, and $\alpha \in (0, 1-1/\beta)$.

We denote by $\Delta^n_{i,k}X$ the kth order increments of the process X over time-lag $1/n$:

$$\Delta^n_{i,1}X := X_{i/n} - X_{(i-1)/n},$$
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Based on these increments, we consider for a (continuous) function f the variation functional

$$V(f)_t^n = \sum_{i=k}^{[nt]} f(a_n \Delta_{i,k}^n X),$$

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Example: For $f(x) = |x|^p$, $p > 0$ the functional $V(f)_t^n$ is the realised power variation of X.

We derive first order and second order limit theorems for $V(f)_t^n$.

Throughout this talk:

$$X_t = \int_{-\infty}^{t} g(t-s) - g_0(s)\,\mathrm{d}L_s,$$

where $(L_t)_{t\in\mathbb{R}}$ is a symmetric pure jump Lévy process with Lévy measure ν.

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- $\beta \in [0,2)$: Blumenthal-Getoor index of L, defined as

$$\beta := \inf\left\{r \geq 0 : \int_{-1}^{1} |x|^r \nu(\mathrm{d}x) < \infty\right\}.$$

If L is stable Lévy process, β is the index of stability.

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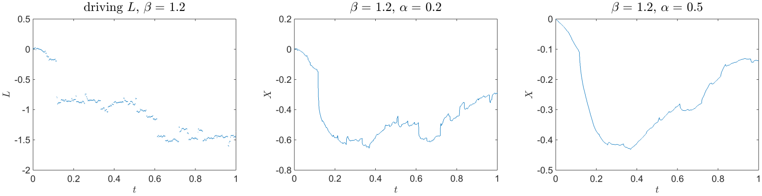
The limiting behavior of $V(f)_t^n$ depends on α, β and f. We obtain three different regimes with different limits and convergence rates.

Examples of LDMA processes



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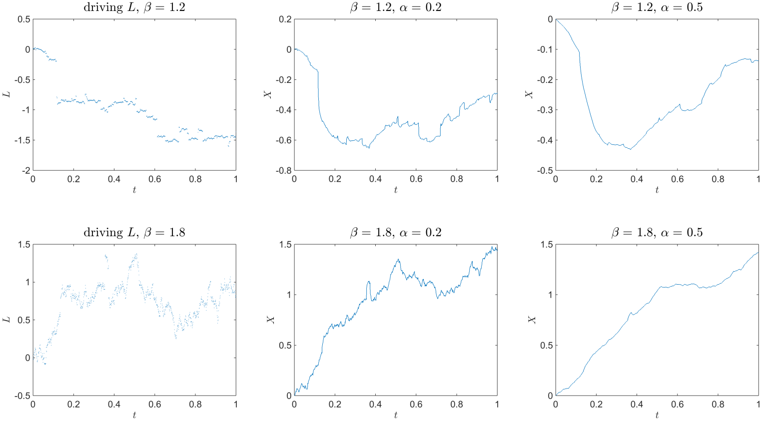

Some previous and related work:

- Basse-O'Connor, Lachiéze-Rey, and Podolskij (2016): First and second order limit theory for power variation of LDMAs driven by a pure jump Lévy process.
- Basse-O'Connor, Heinrich, and Podolskij (2017): First order limit theory for power variation of Lévy semi-stationary processes driven by a pure jump Lévy process, that is for the model

$$X_t = \int_{-\infty}^{t} \{g(t-s) - g_0(-s)\}\sigma_s dL_s.$$

- Barndorff-Nielsen, Corcuera, and Podolskij (2009, 2011): First and second order limit theory for power variation of Brownian semi-stationary processes (the model driven by Brownian motion).

Section 2

First order limit theorems

Theorem 1 (Basse-O'Connor, H., Podolskij)

(i) Let $k > \alpha$ and assume $f(0) = 0$ and that $f \in \mathcal{C}^p$ for some $p > \beta \vee \frac{1}{k-\alpha}$. We obtain the $\mathcal{F}$-stable convergence

$$\sum_{i=k}^{[tn]} f(n^\alpha \Delta_{i,k}^n X) \xrightarrow{\mathcal{L}-s} \sum_{m: T_m \in [0,t]} \sum_{l=0}^{\infty} f\big(\Delta L_{T_m} h_k(l + U_m)\big),$$

where $(U_m)_{m\geq 1}$ is a sequence of independent and $\mathcal{U}([0,1])$-distributed random variables, defined on an extension of the original probability space, independent of L. The function h_k is defined as

$$h_k(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^\alpha, \qquad x \in \mathbb{R}.$$

$f \in \mathcal{C}^p$ if f is $[p]$-times continuously differentiable and $f^{([p])}$ is locally Hölder continuous of order $p - [p]$.

Theorem 1 (Basse-O'Connor, H., Podolskij)

(ii) Suppose that $(1 \vee \beta)(k - \alpha) < 1$. Let f be continuous and assume that $f(x) \leq C(1 \vee |x|^q)$ for some q with $q(k - \alpha) < 1$, and some finite constant C. We have that

$$\frac{1}{n} \sum_{i=k}^{[nt]} f(n^k \Delta_{i,k}^n X) \xrightarrow{\mathbb{P}} \int_0^t f(F_u)\, du,$$

where $F_u = \int_{-\infty}^u g^{(k)}(u - s)\, dL_s$.

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Proof: For $(1 \vee \beta)(k-\alpha) < 1$, the sample paths of X are almost surely k times absolutely continuous with k-th derivative F, see Braverman and Samorodnitsky (1998). It follows by the mean value theorem that

$$n^{-1}\sum_{i=1}^{[nt]} f(n^k \Delta_{i,k}^n X) \approx n^{-1}\sum_{i=1}^{[nt]} f(F_{\frac{i-1}{n}}), \qquad \text{for large } n.$$

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(iii) Suppose that L is a symmetric β-stable Lévy process. Assume that $H := \alpha + 1/\beta < k$ and let f be continuous with $\mathbb{E}[|f(L_1)|] < \infty$. Then we obtain

$$\frac{1}{n} \sum_{i=k}^{[nt]} f(n^H \Delta_{i,k}^n X) \xrightarrow{\mathbb{P}} \mathbb{E}[f(S)],$$

where S is a symmetric β-stable random variable.

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Proof: Let $Y_t := \int_{-\infty}^t (t-s)_+^\alpha - (-s)_+^\alpha \, dL_s$ denote the linear fractional stable motion driven by L, which is stationary and mixing. It holds that

$$\frac{1}{n}\sum_{i=1}^{[nt]} f(n^{\alpha+1/\beta}\Delta_{i,k}^n X) \approx \frac{1}{n}\sum_{i=1}^{[nt]} f(n^{\alpha+1/\beta}\Delta_{i,k}^n Y)$$

$$\stackrel{\mathrm{d}}{=} \frac{1}{n}\sum_{i=1}^{[nt]} f(\Delta_{i,k}^1 Y) \xrightarrow{\mathbb{P}} \mathbb{E}[f(Y_1)].$$

Functional convergence

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  - $(V(f)^n_t)_{t\geq 0}$ converges stably w.r.t. M_1 and M_2 under certain additional assumptions on f, e.g. if f is nonnegative.
  - For many functions, such as for example $f(x) = \sin(x)$, $(V(f)^n_t)_{t\geq 0}$ does not converge stably with respect to either of the 4 Skorokhod-topologies.

Section 3

Second order limit theorems

When L is symmetric β-stable, we derive second order asymptotic results of the form

$$n^{\gamma}\left(n^{-1}\sum_{i=k}^{n}\left\{f(n^{H}\Delta_{i,k}^{n}X)-\mathbb{E}[f(n^{H}\Delta_{i,k}^{n}X)]\right\}\right)\xrightarrow{\mathcal{L}} S,$$

where $H=\alpha+1/\beta$. Here, S is a stable random variable and $\gamma>0$.

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- $n^{H}\Delta_{i,k}^{n}X\overset{\mathrm{d}}{\approx}\Delta_{i,k}^{1}Y$ where Y is linear fractional stable motion.
- Close connection to second order limit theorems of discrete time moving averages driven by stable noise, e.g. Ho and Hsing (1997); Pipiras and Taqqu (2003); Surgailis (2004).

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- Close connection to second order limit theorems of discrete time moving averages driven by stable noise, e.g. Ho and Hsing (1997); Pipiras and Taqqu (2003); Surgailis (2004).
- Two cases may occur: If $(k-\alpha)\beta>2$, a central limit theorem applies, if $(k-\alpha)\beta<2$ the limiting variable S has stability index $(k-\alpha)\beta$ and the convergence rate is $\gamma=1-\frac{1}{(k-\alpha)\beta}$.

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$$\Phi_\rho(x) := \mathbb{E}[f(x + \rho S) - f(\rho S)].$$

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Conditions on Φ_ρ

For all ρ in a compact subset $K \subset \mathbb{R}_+$ there is a constant $C = C_K$ such that

1. $|\Phi_\rho(x) - \Phi_\rho(y)|$
$$\leq C\Big\{(1 \wedge |x| + 1 \wedge |y|)|x - y| 1_{\{|x-y| \leq 1\}} + |x - y|^p 1_{\{|x-y|>1\}}\Big\}$$
2. Φ_ρ is twice differentiable and both derivatives are bounded.

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The two conditions imply in particular that $\Phi'_\rho(0) = 0$ for all $\rho > 0$.

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Intuitively speaking, this is satisfied whenever f is even around 0 (or $f'(0) = 0$), and grows slower than $|x|^p$ for some $p \in (0, \beta/2)$, as $|x| \to \infty$.

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- negative power functions $f(x) = |x|^q$ with $q \in (-1/2, 0)$,
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Examples:

- power functions $f(x) = |x|^p$ with $p \in (0, \beta/2)$,
- negative power functions $f(x) = |x|^q$ with $q \in (-1/2, 0)$,
- bounded functions that are continuously differentiable at 0 with $f'(0) = 0$, e.g. $f(x) = \cos(x)$ or $f(x) = 1_{[a,\infty)}(x)$ for $a \neq 0$,
- $f(x) = \log(|x|)$,

Second order Asymptotics

Theorem 5, (Basse-O'Connor, H., Podolskij)

Let L be a symmetric β-stable Lévy process and previously discussed conditions on f be satisfied. Set $H = \alpha + \frac{1}{\beta}$.

(i) Suppose that $\alpha \in (0, k - 2/\beta)$, then it holds that

$$\sqrt{n}\left(n^{-1}\sum_{i=k}^{n}\{f(n^H\Delta_{i,k}^n X) - \mathbb{E}[f(n^H\Delta_{i,k}^n X)]\}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\eta^2).$$

(ii) Suppose that $\alpha \in (k - 2/\beta, k - 1/\beta)$. It holds that

$$n^{1-\frac{1}{(k-\alpha)\beta}}\left(n^{-1}\sum_{i=k}^{n}\{f\left(n^H\Delta_{i,k}^n X\right) - \mathbb{E}\left[f\left(n^H\Delta_{i,k}^n X\right)\right]\}\right) \xrightarrow{\mathcal{L}} S,$$

where S is a $(k-\alpha)\beta$-stable random variable with location parameter 0.

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$$\sum_{r=k}^{n} \left(f(\Delta_{i,k}^1 Y) - \sum_{j=1}^{\infty} \mathbb{E}[f(\Delta_{i,k}^1 Y) \,|\, \mathcal{F}_{r-j}^1] \right) \xrightarrow{L^2} 0,$$

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- The proof is completed by deriving

$$c_+ := \lim_{x \to \infty} x^{(k-\alpha)\beta} \mathbb{P}[Z_1 > x], \quad \text{and} \quad c_- := \lim_{x \to -\infty} |x|^{(k-\alpha)\beta} \mathbb{P}[Z_1 < x].$$

Additional remarks:

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- Main difference to limit theory of discrete time moving averages is that the increments $n^H \Delta_{i,k}^n X$ are symmetric β-stable distributed with scale parameter ρ_n depending on n. The sequence ρ_n converges to the scale parameter of the associated linear fractional stable motion.

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- Koul and Surgailis (2001): When $\Phi'(0) \neq 0$, the discrete time statistic $\sum_{i=k}^{n} f(\Delta_{i,k}^1 X)$ is asymptotically α-stable.

Application:
Estimation of $H = \alpha + 1/\beta$ by taking quotients of power variations based on different frequencies. When L is β-stable the power variation functional $V(p)^n_t$ satisfies by Theorem 1 (iii)

$$\frac{\sum_{i=2}^{n} |X_{\frac{i}{n}} - X_{\frac{i-2}{n}}|^p}{\sum_{i=1}^{n} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^p} \xrightarrow{\mathbb{P}} 2^{pH},$$

for all $p < \beta$, and similarly for higher order increments.

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- Theorem 2 implies the asymptotic normality of the estimator for sufficiently high order of increments k.

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