Superpositions and Products of Ornstein-Uhlenbeck Type Processes: Intermittency and Multifractality

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Abstract

Superpositions of stationary processes of Ornstein-Uhlenbeck (supOU) type have been introduced by Barndorff-Nielsen. We consider the constructions producing processes with long-range dependence and infinitely divisible marginal distributions.

We consider additive functionals of supOU processes that satisfy the property referred to as intermittency.

We investigate the properties of multifractal products of supOU processes. We present the general conditions for the L_q convergence of cumulative processes and investigate their q-th order moments and Rényi functions. These functions are nonlinear, hence displaying the multifractality of the processes.

We also establish the corresponding scenarios for the limiting processes, such as log-normal, log-gamma, log-tempered stable or log-normal tempered stable scenarios.

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OU type process

- **OU Type process** is the unique strong solution of the SDE:

$$dX(t) = -\lambda X(t)dt + dZ(\lambda t)$$

  where $\lambda > 0$, $\{Z(t)\}_{t\geq 0}$ is a (non-decreasing, for this talk) Lévi process, and an initial condition X_0 is taken to be independent of $Z(t)$. Note, in general Z_t doesn't have to be a non-decreasing Lévy process.
- For properties of OU type processes and their generalizations see Mandrekar & Rudiger (2007), Barndorff-Nielsen (2001), Barndorff-Nielsen & Stelzer (2011).
- The a.s. unique solution is of the form:

$$X(t) = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(s-t)}dZ(\lambda s), \qquad t \geq 0.$$

Infinite superposition of OU type process

Let $\{X^{(k)}(t)\}_{k\geq 1}$ be a sequence of independent OU type processes. Define an infinite superposition as:

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t)$$

Infinite superpositions are well defined under the following assumption:

$$\textbf{(A):} \quad \sum_{k=1}^{\infty} \mathbb{E}X^{(k)}(t) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \mathrm{Var}X^{(k)}(t) < \infty.$$

Infinite superposition of OU type process

Assumption (B): **(B1):** The self decomposable distributions of $X^{(k)}$ have all moments and cumulants of all orders.

(B2): The marginal distributions of $X^{(k)}$ are closed under convolution with respect to at least one parameter δ_k, and all cumulants are proportional to that parameter.

Covariance structure of infinite supOU processes

The covariance function is of the form

$$R_{X_\infty}(t) = cov(X_\infty(0), X_\infty(t)) = \sum_{k=1}^{\infty} \mathrm{Var}(X^{(k)}(t)) e^{-\lambda_k t},$$

Assumption **(B)** implies that: $\mathrm{Var}(X_t^{(k)}) = \delta_k C$, where C is a constant, reflects parameters of the marginals of $X^{(k)}$. For the specific choice of parameters $\delta_k = k^{-(1+2(1-H))}$, $1/2 < H < 1$, and $\lambda_k = \lambda/k, \lambda > 0$, we get

$$R_{X_\infty}(t) = C_2 \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\lambda t/k}.$$

The covariance function $R_{X_\infty}(t)$ is not integrable for the chosen parameters δ_k and λ_k, i.e. the infinite supOU exhibits **long-range dependence (LRD)**, as we show next.

Covariance structure of infinite supOU processes

For the infinite superposition of OU type processes that satisfy condition (B) and condition (A), the covariance function of $X_\infty(t)$, with specific $\lambda^{(k)} = \lambda/k$ and $\delta_k = k^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, can be written as

$$R_{X_\infty}(t) = \frac{L(t)}{t^{2(1-H)}}, \quad t > 0$$

where function L is a slowly varying at infinity function. Observe that, for given H, the covariance function is not integrable at infinity, hence infinite supOU process is long-range dependent (LRD).

Scaling function and intermittency

For a process $\{Y(t)\}_{t\geq 0}$, denote $\overline{q} = \sup\{q > 0 : E|Y(t)|^q < \infty, \forall t\}$ and assume that for the function below limit exists and is finite for every $q \in [0, \overline{q})$. Then define a *scaling function* at point $q \in [0, \overline{q})$ as:

$$\tau(q) = \lim_{t\to\infty} \frac{\log E|Y(t)|^q}{\log t}, q \in [0, \overline{q}).$$

We say a stochastic process $\{Y_t\}_{t\geq 0}$ is intermittent if there exist two points $p, r \in (0, \overline{q})$ such that $\tau(p)/p < \tau(r)/r$.

Note: Scaling function is convex and non-decreasing function. The function $\tau(q)/q$ is always non-decreasing, and what makes the process Y_t intermittent is the existence of points of strict increase.

Example of a non-intermittent sequence

Slow moment growth: Denote the sum of positive, independent and identically distributed (iid) random variable with finite moments: $S_n = \sum_{i=1}^{n} \xi_i$. Then its q-th moment grows as: $E(S_n)^q \sim n^q (E\xi_1)^q$, and the scaling function of S_n is of the form:

$$\tau(q) = \lim_{t\to\infty} \log \mathbb{E}|S(n)|^q / \log n = \lim_{t\to\infty} (q \log n + q \log E\xi_1) / \log n = q.$$

Remark: Intuitively, the scaling function shows how q-th moment of the process $\{Y(t)\}_{t\geq 0}$ asymptotically behaves as a function of time:

$$\mathbb{E}|Y(t)|^q \sim t^{\tau(q)}, t \to \infty.$$

Intermittency as multifractality

Theorem. Let $\{X_\infty(t), t \geq 0\}$ be a non-Gaussian (discrete)

supOU process such that the cumulant function $\kappa_X(\zeta)$ of the OU process $\{X(t), t \geq 0\}$ is analytic in the neighborhood of the origin and $\kappa_X^{(1)} = 0$ and $\kappa_X^{(2)} \neq 0$. If τ_Y is the scaling function of

$$Y = \{Y(t) = \sum_{i=1}^{[t]} [X_\infty(i) - \mathbb{E}X_\infty(i)],\ t \geq 0\},$$

then for every $q \geq q^*$

$$\tau_Y(q) = q - \alpha,$$

where q^* is the smallest even integer greater than 2α, $\alpha = 2(1 - H)$. Thus Y is intermittent.

Note that for self-similar processes Y

$$\tau_Y(q) = qH,$$

Moreover, the function

$$\frac{\tau_Y(q)}{q} = 1 - \frac{2(1-H)}{q} = H(q)$$

is strictly increasing in q :

$$\tau_Y(1) < \frac{\tau_Y(2)}{2} < \frac{\tau_Y(3)}{3} < ... < \frac{\tau_Y(q)}{q} <$$

The term $H(q)$ in the exponent of the asymptotic behavior of the q-th cumulant of $Y(tN)$:

$$\kappa_Y^{(q)}(Nt) = C_q L(N)[Nt]^{qH(q)}(1 + o(1)), \quad N \to \infty,$$

that is supOU processes are intermittent.

Thus, intremittency can be interpreted as special case of multifractality defined below.

Multifractal products of stochastic processes

Kahane (1985, 1987), Mennersalo, Norros and Reidi (2002): for an independent copies $\Lambda_0(s), ..., \Lambda_n(s)$ of the mother process $\Lambda(s)$:

$$A(t) = \lim_{n\to\infty} \int_0^t [\Lambda_0(s) \cdot \cdots \cdot \Lambda_n(s)] ds, t \in T \in \mathbb{R}_+$$

where $\{A(t), t \geq 0\}$ is maltifractal, that is for some non-linear function $\zeta(q), q \in Q \subseteq \mathbb{R}$

$$\mathbb{E}A^q(t) \sim t^{\zeta(q)},$$

for example (Kolmogorov's lognormal scenario):

$$\zeta(q) = -aq^2 + (a+1)q, \qquad a > 0.$$

For a monofractal processes $\zeta(q)$ is a linear function, for example

$$\zeta(q) = qH,$$

where $H \in (0, 1)$ is the Hurst exponent (FBM, α-stable motion, and α-stable subordinator and its inverse).

Examples:

1) Binomial cascade: $\Lambda_n(s)$ is constant on dyadic intervals;
2) Martingale de Mandelbrot: $\mathbb{E}\Lambda_n(s) = 1$;
3) Stationary cascade: $\Lambda_n(s)$ is stationary:
   i) conservation

$$\mathbb{E}\Lambda_n(s) = 1$$

   ii) "self-similarity"

$$\Lambda_n(s) =^d \Lambda_1(sb^n), b > 1.$$

We consider the case

$$\Lambda(t) = e^{X_\infty(t) - c_X},$$

where $X_\infty(t)$ is supOU stationary process with marginal distributions:

$Gaussian, \ Gamma, IG, NIG, TS, NTS, VG, ...$

Product process

The conditions $\mathbf{A}'$-$\mathbf{A}'''$ yield

$$\begin{aligned}\mathbb{E}\Lambda_b^{(i)}(t) &= M(1) = 1;\\ \mathrm{Var}\Lambda_b^{(i)}(t) &= M(2) - 1 = \sigma_\Lambda^2 < \infty;\end{aligned}$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \; b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = \exp\left\{ \sum_{i=0}^{n} X^{(i)}(tb^i) \right\}, t \in [0, 1],$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s) ds, \quad n = 0, 1, 2, \ldots, t \in [0, 1],$$

where $X^{(i)}(t), i = 0, ..., n, ...,$ are independent copies of a stationary process $X(t), t \geq 0$.

Random measures

We also consider the corresponding positive random measures defined on Borel sets B of $\mathbb{R}_+$:

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0, 1, 2, \ldots$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $\mathbb{R}_+$, it holds that $\lim_{n\to\infty} \mu_n(B_j) = \mu(B_j)$ for all j with probability one.

Random measures continued

The almost sure convergence of $A_n(t)$ in countably many points of $\mathbb{R}_+$ can be extended to all points in $\mathbb{R}_+$ if the limit process $A(t)$ is almost surely continuous. In this case, $\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987), there are two extreme cases:

(i) $A_n(t) \to A(t)$ in $\mathcal{L}_1$ for each given t, in which case $A(t)$ is not almost surely zero and and is said to be fully active (non-degenerate) on $\mathbb{R}_+$;

(ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on $\mathbb{R}_+$.

Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987). The condition for complete degeneracy is detailed in Theorem 3 of Kahane (1987).

Conditions

We introduce the following conditions:

A′ Let $\Lambda(t), \;\; t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $\mathbb{E}\Lambda(t) = 1$. We call this process the mother process and consider the following setting:

A″ Let $\Lambda(t) = \Lambda^{(i)}, \;\; i = 0, 1, ...$ be independent copies of the mother process Λ, and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$:

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \ldots,$$

where the scaling parameter $b > 1$, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions.

Conditions continued

Moreover, in the examples, the stationary mother process satisfies the following conditions:

A'''. Let $\Lambda(t) = \exp\{X(t)\}$, $t \in \mathbb{R}_+$, where $X(t)$ is a strictly stationary process, such that there exist a marginal probability density function $\pi(x)$ and a bivariate probability density function $p(x_1, x_2; t_1 - t_2)$. Moreover, we assume that the moment generating function

$$M(\zeta) = \mathbb{E}\exp\{\zeta X(t)\}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathbb{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$$

exist.

The Rényi function

The Rényi function of a random measure μ, also known as the deterministic partition function, is defined for $t \in [0,1]$ as

$$T(q) = \liminf_{n\to\infty} \frac{\log \mathbb{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right)}{\log \left|I_k^{(n)}\right|}$$

$$= \liminf_{n\to\infty} \left(-\frac{1}{n}\right) \log_2 \mathbb{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right),$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $\quad k = 0, 1, \ldots, 2^n - 1$, $\left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

We establish convergence

$$A_n(t) \stackrel{\mathcal{L}_q}{\to} A(t), \quad n \to \infty.$$

For the limiting process we show that for some constants $\overline{C}$ and $\underline{C}$,

$$\underline{C} t^{q - \log_b \mathbb{E}\Lambda^q(t)} \leqslant \mathbb{E}A^q(t) \leqslant \overline{C} t^{q - \log_b \mathbb{E}\Lambda^q(t)},$$

which will be written as

$$\mathbb{E}A^q(t) \sim t^{q - \log_b \mathbb{E}\Lambda^q(t)}.$$

This allows us to find the scaling function

$$\varsigma(q) = q - \log_b \mathbb{E}\Lambda^q(t) = q - \log_b M(q).$$

As is shown in Leonenko and Shieh (2012) for $q \in [1, 2]$ there is a connection between Rényi function and the scaling function given by

$$T(q) = \varsigma(q) - 1.$$

Remark

The multifractal formalism for random cascades and other multifractal processes can be stated in terms of the Legendre transform of the Rényi function:

$$T^*(\alpha) = \min_{q\in\mathbb{R}} \left(q\alpha - T(q)\right).$$

In fact, let $f(\alpha)$ be the Hausdorff dimension of the set

$$C_\alpha = \left\{ t \in [0,1] : \lim_{n\to\infty} \frac{\log \mu\left(I_k^{(n)}(t)\right)}{\log \left|I_k^{(n)}\right|} = \alpha \right\},$$

where $I_k^{(n)}(t)$ is a sequence of intervals $I_k^{(n)}$ that contain t. The function $f(\alpha)$ is known as the singularity spectrum of the measure μ, and we refer to μ as a multifractal measure if $f(\alpha) \neq 0$ for a continuum of α.

Martingales

Consider the cumulative process $A_n(t)$ For fixed t, the sequence $\{A_n(t), \mathcal{F}_n\}_{n=0}^{\infty}$ is a martingale. It is well known that for $q > 1$, $\mathcal{L}_q$ convergence is equivalent to the finiteness of

$$\sup_n \mathbb{E} A_n^q(t) < \infty.$$

Condition for Log-normal scenario

B$'$. Consider a mother process of the form

$$\Lambda(t) = \exp\left\{X(t) - \frac{1}{2}\sigma_X^2\right\},$$

where $X(t)$, $t \in [0,1]$ is a zero-mean Gaussian, measurable, separable stochastic process with covariance function

$$R_X(\tau) = \sigma_X^2 \mathrm{Corr}(X(t), X(t+\tau))$$

Moment generating functions

Under condition $\mathbf{B}'$, we obtain the following specifications of the moment generating functions:

$$\begin{aligned} M(\zeta) &= \mathbb{E}\exp\left\{\zeta\left(X(t)-\frac{1}{2}\sigma_X^2\right)\right\} = e^{\frac{1}{2}\sigma_X^2(\zeta^2-\zeta)}, \quad \zeta\in\mathbb{R}^1, \\ M(\zeta_1,\zeta_2;t_1-t_2) &= \mathbb{E}\exp\left\{\zeta_1\left(X(t_1)-\frac{1}{2}\sigma_X^2\right)+\zeta_2\left(X(t_2)-\frac{1}{2}\sigma_X^2\right)\right\} \\ &= \exp\left\{\frac{1}{2}\sigma_X^2\left[\zeta_1^2-\zeta_1+\zeta_2^2-\zeta_2\right]+\zeta_1\zeta_2R_X(t_1-t_2)\right\}, \\ &\quad \zeta_1,\zeta_2\in\mathbb{R}^1, \end{aligned}$$

where $\sigma_X^2\in(0,\infty)$.

Moment generating functions continued

It turns out that, in this case,

$$\begin{aligned} M(1) &= 1; \quad M(2) = e^{\sigma_X^2}; \qquad \sigma_\Lambda^2 = e^{\sigma_X^2} - 1; \\ \mathrm{Cov}(\Lambda(t_1), \Lambda(t_2)) &= M(1, 1; t_1 - t_2) - 1 \\ &= e^{R_X(t_1 - t_2)} - 1 \\ &\approx R_X(t_1 - t_2) \end{aligned}$$

and

$$\log_b \mathbb{E}\Lambda(t)^q = \frac{(q^2 - q)\sigma_X^2}{2 \log b}, \quad q > 0.$$

Theorem

Suppose that condition $\mathbf{B}'$ holds with the correlation function

$$\mathrm{Corr}(X(t), X(t+\tau)) \leq C\tau^{-\alpha}, \quad \alpha > 0,$$

for sufficiently large τ, and for some $a > 0$,

$$1 - \mathrm{Corr}(X(t), X(t+\tau)) \leq C\left|\tau\right|^{a},$$

for sufficiently small τ. Assume that

$$b > \exp\left\{q^{*}\sigma_X^2/2\right\},$$

where $q^{*} > 0$ is a fixed integer. Then the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^{j}\right) ds, t \in [0,1]$$

converge in $\mathcal{L}_q, 0 < q \leq q^{*}$ to the stochastic process $A(t), t \in [0,1]$, as $n \to \infty$, such that

$$\mathbb{E}A(t)^q \sim t^{-aq^2+(a+1)q}, q \in [0, q^{*}],$$

Theorem continued

The Rényi function is given by

$$T(q) = -aq^2 + (a+1)q - 1, q \in (0, q^*) \cap [1, 2],$$

where

$$a = \frac{\sigma_X^2}{2 \log b}.$$

Theorem continued

Moreover, if

$$\mathrm{Corr}(X(t), X(t+\tau)) = \frac{L(\tau)}{|\tau|^{\alpha}}, \alpha > 0,$$

where L is a slowly varying at infinity function, then

$$\mathrm{Var}A(t) \geqslant t^{2-\alpha}\sigma_X^2 \int_0^1 \int_0^1 \frac{L(t\,|u-v|)dudw}{L(t)\,|u-w|^{\alpha}}, 0 < \alpha < 1,$$

and

$$\mathrm{Var}A(t) \geqslant 2t\sigma_X^2 \int_0^t (1-\frac{\tau}{t})\frac{L(\tau)}{|\tau|^{\alpha}}d\tau, \alpha \geq 1.$$

Remark

We interpret the inequality as a form of long-range dependence of the limiting process in the following sense: one can replace the interval $[0, 1]$ into more general interval $[0, t]$, and for a large t we have the following:

$$\begin{aligned}\operatorname{Var} A(t) &\geq \lim_{t\to\infty} \int_0^t \int_0^t \frac{L(|u-v|)dudw}{|u-w|^\alpha} \\ &= \lim_{t\to\infty} t^{2-\alpha}\sigma_X^2 \int_0^1 \int_0^1 \frac{L(t\,|u-v|)dudw}{L(t)\,|u-w|^\alpha},\end{aligned}$$

and

$$\lim_{t\to\infty} \int_0^1 \int_0^1 \frac{L(t\,|u-v|)dudw}{L(t)\,|u-w|^\alpha} = \int_0^1 \int_0^1 \frac{dudw}{|u-w|^\alpha} = \frac{2}{(1-\alpha)(2-\alpha)},$$

for $0 < \alpha < 1$.

LRD sup OU processes

We are going to consider an infinite superposition of the OU processes, which corresponds to $m = \infty$, that is now

$$X_\infty(t) = \sum_{j=1}^{\infty} X_j(t),$$

assuming that

$$\sum_{j=1}^{\infty} \mathbb{E} X_j(t) < \infty, \sum_{j=1}^{\infty} \mathrm{Var} X_j(t) < \infty,$$

LRD sup OU processes

In this case

$$R_\infty(t) = \sum_{j=1}^{\infty} \sigma_j^2 \exp\{-\lambda_j |t|\},$$

and if we assume that for some $\delta_j > 0$

$$\mathbb{E}X_j(t) = \delta_j C_1, \mathrm{Var} X_j(t) = \sigma_j^2 = \delta_j C_2, \delta_j = j^{-(1+2(1-H))}, \frac{1}{2} < H < 1,$$

where the constants $C_1 \in \mathbf{R}$ and $C_2 > 0$ represent some other possible parameters, then

$$\mathbb{E}X_\infty(t) = C_1 \sum_{j=1}^{\infty} \delta_j = C_1 \zeta(1 + 2(1 - H)) < \infty,$$

LRD sup OU processes

We are going to make an additional assumption that there exist parameters δ_j such that

$$\mathbb{E}e^{\zeta X_j(0)} = \mathbb{E}e^{\zeta \delta_j Y}$$

for some random variable Y. The sum

$$\sum_{j=1}^{\infty} \delta_j < \infty$$

must be finite. We obtain

$$R_\infty(t) = C_2 \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\},$$

for some $C_2 > 0$. This approach allows also to treat the case of several parameters.

LRD geometric sup OU processes

Define the mother process as geometric process

$$\Lambda(t) = e^{X_\infty(t) - c_X}, c_X = \log \mathbb{E} e^{X_\infty(0)}, M(\zeta) = \mathbb{E} e^{\zeta(X_\infty(0) - c_X)},$$

where $X_\infty(t), t \in \mathbb{R}$, is the infinite superposition process.
Denote

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \mathbb{E} \exp\{\zeta_1 X_\infty(t_1) + \zeta_2 X_\infty(t_2)\}.$$

Theorem

Let $X_\infty(t), t \in \mathbb{R}_+$ be an infinite superposition of OU-type stationary processes. Assume that the Lévy measure ν of the random variable $X_{\text{sup}}(t)$ satisfies the condition that for a positive integer $q^* \in \mathbf{N}$,

$$\int_{|x|\geq 1} x e^{q^* x} \nu(dx) < \infty.$$

Then, for any fixed b such that

$$b > \left\{ \frac{M(q^*)}{M(1)^{q^*}} \right\}^{\frac{1}{q^*-1}},$$

Theorem continued

the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^j\right) ds, t \in [0,1]$$

converge in $\mathcal{L}_q$ to the stochastic process $A(t) \in \mathcal{L}_q, t \in [0,1]$, as $n \to \infty$. The limiting process $A(t)$ satisfies

$$\mathbb{E}A^q(t) \sim t^{q - \log_b \mathbb{E}\Lambda^q(t)}, \quad q \in [0, q^*].$$

Theorem continued

The Rényi function is given by

$$T(q) = q - 1 - \log_b \mathbb{E}\Lambda^q(t), \quad q \in [0, q^*] \cap [1, 2], \; t \in [0, 1].$$

In addition,

$$\begin{aligned} \mathrm{Var} A(t) &\geqslant \int_0^t \int_0^t M(\zeta_1, \zeta_2; s_1 - s_2) ds_1 ds_2 \\ &\approx \int_0^t \int_0^t R_X(s_1 - s_2) ds_1 ds_2, \end{aligned}$$

where $M(\zeta_1, \zeta_2; s_1 - s_2)$ is the bivariate moment generating function.

Log-gamma scenario

The log-gamma multifractal scenario is well-known in the theory of turbulence and multiplicative cascades (Saito 1992). In this section, we propose a stationary version of the log-gamma scenario with LRD.

We will use a stationary OU type process $X(t), t \in \mathbb{R}_+$, with marginal gamma distribution $\Gamma(\beta, \alpha)$. It is known that the gamma distribution with the moment generating function

$$\mathbb{E}\exp\{\zeta X(t)\} = \left(1 - \frac{\zeta}{\alpha}\right)^{-\beta}, \ \zeta < \alpha, \alpha > 0, \beta > 0,$$

is self-decomposable. The Lévy triplet is of the form $(0, 0, \nu)$, where

$$\nu(du) = \frac{\beta e^{-\alpha u}}{u} \mathbf{1}_{[0,\infty)}(u) du.$$

The covariance function is then

$$r_X(t) = (\beta/\alpha^2) \exp(-\lambda |t|).$$

One can construct supOU processes $X_\infty(t)$, where $X_j(t), j = 1, ...,$ are independent stationary processes with marginals $\Gamma(\beta_j, \alpha), j = 1, 2...$ Then

$$X_\infty(t) = \sum_{j=1}^{\infty} X_j(t) \sim \Gamma(\sum_{j=1}^{\infty} \beta_j, \alpha),$$

assuming $\beta_\infty = \sum_{j=1}^{\infty} \beta_j < \infty$.
Consider a mother process of the form

$$\Lambda(t) = \exp\left(X_\infty\left(t\right) - c_X\right), \quad c_X = \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^{\beta_\infty}}, \alpha > 1.$$

We obtain the following moment generating functions:

$$M(\zeta) = \mathbb{E}\exp\left(\zeta\left(X_\infty(t) - c_X\right)\right) = \frac{e^{-c_X\zeta}}{\left(1-\frac{\zeta}{\alpha}\right)^{\beta_\infty}}, \ \zeta < \alpha, \alpha > 1,$$

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E}\exp\left\{\zeta_1\left(X_\infty(t_1) - c_X\right) + \zeta_2\left(X_\infty(t_2) - c_X\right)\right\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathbb{E}\exp\left\{\zeta_1 X_\infty(t_1) + \zeta_2\left(X_\infty(t_2)\right)\right\}, \end{aligned}$$

where

$$\log\mathbb{E}\exp\{\zeta_1 X_\infty(t_1)+\zeta_2 X_\infty(t_2)\} = \sum_{j=1}^{\infty}\log\mathbb{E}\exp\{\zeta_1 X_j(t_1)+\zeta_2 X_j(t_2)\},$$

and

$$\begin{aligned} \log\mathbb{E}\exp\Big(\zeta_1 X_j(t_1) \quad &+ \quad \zeta_2\left(X_j(t_2)\right)\Big) = \\ &= \int_{\mathbb{R}} \frac{\beta_j \sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j - s)}{\alpha - \sum_{j=1}^{2}\zeta_j e^{-\lambda(t_j-s)}\mathbf{1}_{[0,\infty)}(t_j - s)} ds. \end{aligned}$$

It turns out that, in this case,

$$\log_b \mathbb{E}\Lambda(t)^q = \frac{1}{\log b_\infty}\left(-q\log\frac{1}{\left(1-\frac{1}{\alpha}\right)^{\beta_\infty}} - \beta_\infty \log\left(1-\frac{q}{\alpha}\right)\right),$$

and

$$\int_{|u|\geq 1} u e^{q^* u}\nu(du) = \frac{\alpha^{\beta_\infty}\beta_\infty}{\Gamma(\beta_\infty)}\int_1^\infty e^{q^* u} e^{-\alpha u} du < \infty, q^* < \alpha.$$

We formulate the following

Theorem. Let $X(t)$ be a stationary gamma supOU stochastic process and let

$$Q = \{q : 0 < q < q^* < \alpha, \alpha > 2, \beta_\infty > 0\},$$

where q^* is a fixed integer. Then, for any

$$b > \left[\left(1 - \frac{1}{\alpha}\right)^{\beta_\infty q^*} / \left(1 - \frac{q^*}{\alpha}\right)^{\beta_\infty} \right]^{\frac{1}{q^*-1}},$$

the stochastic processes $A_n(t)$ defined as in condition $\mathbf{B}'''$ converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ as $n \to \infty$, such that $A(t) \in \mathcal{L}_q$ and

$$\mathbb{E}A(t)^q \sim t^{\varsigma(q)},$$

where the scaling function $\varsigma(q)$ is given by

$$\varsigma(q) = q\left(1 + \frac{1}{\log b}\log\frac{1}{\left(1 - \frac{1}{\alpha}\right)^{\beta_\infty}}\right) + \frac{\beta_\infty}{\log b}\log\left(1 - \frac{q}{\alpha}\right), q \in Q.$$

Log-NIG scenario

As an example consider a stationary OU-type process $X(t)$ with marginal normal inverse Gaussian distribution $NIG(\alpha, \beta, \delta, \mu)$, which is self-decomposable, and hence infinitely divisible. The moment generating function of $NIG(\alpha, \beta, \delta, \mu)$ is given by the formula:

$$\log M(\zeta) = \mu\zeta + \delta \left[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \zeta)^2} \right], \ |\beta + \zeta| < \alpha,$$

and the set of parameters satisfies the following constraints

$$\delta > 0, 0 \leq |\beta| \leq \alpha, \mu \in \mathbb{R}, \gamma^2 = \alpha^2 - \beta^2.$$

The Lévy triplet is of the form $(a, 0, \nu)$, where

$$\begin{aligned} a &= \mu + 2\pi^{-1}\delta\alpha \int_0^1 \sinh(\beta x) K_1(\alpha x)\, dx, \nu(du) \\ &= \pi^{-1}\delta\alpha |u|^{-1} K_1(\alpha |u|) e^{\beta u} du, \end{aligned}$$

The covariance function is then

$$r_X(t) = (\delta\alpha^2/\gamma^3) \exp(-\lambda |t|).$$

One can construct supOU processes $X_\infty(t)$, where $X_j(t), j = 1, \ldots,$ are independent stationary processes with marginals $NIG(\alpha, \beta, \delta_j, \mu_j), j = 1, 2\ldots$ Then

$$X_\infty(t) = \sum_{j=1}^{\infty} X_j(t) \sim NIG(\alpha, \beta, \sum_{j=1}^{\infty} \delta_j, \sum_{j=1}^{\infty} \mu_j)$$

assuming $\delta_\infty = \sum_{j=1}^{\infty} \delta_j < \infty, \mu_\infty = \sum_{j=1}^{\infty} \mu_j < \infty.$
Consider a mother process of the form

$$\Lambda(t) = \exp\left(X_\infty(t) - c_X\right), \quad c_X = \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^{\beta_\infty}}, \alpha > 1.$$

Consider a mother process of the form

$$\Lambda(t) = \exp\left\{X_\infty(t) - c_X\right\}, c_X = \mu_\infty + \delta_\infty \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2},$$

where $|\beta + 1| < \alpha$. Let $q^* \leq \alpha - |\beta|$ be an integer and put

$$Q = \{q : 0 < q < q^*, |\beta + 1| < \alpha, \mu_\infty \in \mathbb{R}, \ \delta_\infty > 0\}$$

If $b > \exp\left\{-\delta_\infty\sqrt{\alpha^2-\beta^2}+\frac{\delta_\infty\sqrt{\alpha^2-(\beta+q^*)^2}-q^*\delta_\infty\sqrt{\alpha^2-(\beta+1)^2}}{1-q^*}\right\}$, then the statement of main Theorem holds for $q \in Q$ with the scaling function

$$\begin{aligned}\varsigma(q) &= \left(1-\frac{\delta_\infty\left[\sqrt{\beta^2+\gamma^2-(\beta+1)^2}-\gamma\right]}{\log b}\right)q \\ &\quad +\frac{\delta_\infty}{\log b}\sqrt{\beta^2+\gamma^2-(\beta+q)^2}-\frac{\delta\gamma}{\log b}-1.\end{aligned}$$

Log-tempered stable (log-IG) scenario

The cumulant transform of a random variable $X \sim TS(\kappa, \delta, \gamma)$ is of the form.

$$\log \mathbb{E} e^{\zeta X} = \delta\gamma - \delta \left(\gamma^{\frac{1}{\kappa}} - 2\zeta\right)^{\kappa}, 0 < \zeta < \frac{\gamma^{1/\kappa}}{2}.$$

Note that

$$\mathbb{E}X(t) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}}, \mathrm{Var}X(t) = 4\kappa\left(1-\kappa\right)\delta\gamma^{\frac{\kappa-2}{\kappa}}.$$

We will consider a stationary OU type process with marginal distribution $TS(\kappa, \delta, \gamma)$. This distribution is self-decomposable (and hence infinitely divisible) with the Lévy triplet $(a, 0, \nu)$, where

$$\nu(du) = b(u)du, b(u) = 2^{\kappa}\delta\frac{\kappa}{\Gamma\left(1-\kappa\right)}u^{-1-\kappa}e^{-\frac{u\gamma^{1/\kappa}}{2}}, u > 0.$$

Then

$$X_{\infty}(t) = \sum_{j=1}^{\infty} X_j(t) \sim TS(\kappa, \sum_{j=1}^{\infty}\delta_j, \gamma),$$

if $\delta_{\infty} = \sum_{j=1}^{\infty}\delta_j < \infty$.

Consider a mother process of the form

$$\Lambda(t) = \exp\left\{X_\infty(t) - c_X\right\}$$

with

$$c_X = \left[\delta_\infty \gamma - \delta_\infty \left(\gamma^{\frac{1}{\kappa}} - 2\right)^\kappa\right], \gamma > 2^\kappa,$$

and if
$Q = \{q : 0 < q < \frac{\gamma^{1/\kappa}}{2}, \gamma \geq \max\{(2q^*)^\kappa, 4^\kappa\}, \kappa \in (0,1), \delta_\infty > 0\} \cap [0, q^*]$,
where q^* is a fixed integer. Then, for any

$$b > \exp\left\{-\gamma\delta_\infty + \frac{\delta_\infty}{1-q^*}\left(\gamma^{\frac{1}{\kappa}} - 2q^*\right)^\kappa - \frac{q^*}{1-q^*}\delta_\infty\left(\gamma^{\frac{1}{\kappa}} - 2\right)^\kappa\right\},$$

$\gamma \geq \max\{(2q^*)^\kappa, 4^\kappa\}$, the stochastic processes $A_n(t)$ converge in $\mathcal{L}_q$ to the stochastic process $A(t)$ for each fixed $t \in [0,1]$ as $n \to \infty$ such that, $A_q(1) \in \mathcal{L}_q$, for $q \in Q$, and $\mathbb{E}A^q(t) \sim t^{\varsigma(q)}$, where the scaling function $\varsigma(q)$ is given by
$\varsigma(q) = q\left(1 + \frac{\delta_\infty \gamma}{\log b} - \frac{\delta_\infty}{\log b}\left(\gamma^{\frac{1}{\kappa}} - 2\right)\right)^\kappa + \frac{\delta_\infty}{\log b}\left(\gamma^{\frac{1}{\kappa}} - 2q\right)^\kappa - \frac{\delta_\infty \gamma}{\log b}, q \in Q.$

In this particular when $\kappa = 1/2$, we arrive to log-inverse Gaussian scenario where the scaling function is of the form:

$$\varsigma(q) = q\left(1 + \frac{\delta_\infty\left[\gamma - \sqrt{\gamma^2 - 2}\right]}{\log b}\right) + \frac{\delta_\infty}{\log b}\sqrt{\gamma^2 - 2q} - \frac{\gamma\delta_\infty}{\log b}, q \in Q,$$

and

$$Q = \left\{q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2, \delta_\infty > 0\right\} \cap (0, q^*)$$

if

$$b > \exp\left\{-\gamma\delta_\infty - \frac{\delta_\infty}{1 - q^*}\sqrt{\gamma^2 - 2q} - \frac{q^*}{1 - q^*}\delta_\infty\sqrt{\gamma^2 - 2}\right\}$$

and q^* is a fixed integer.

Some other multifractal scenarios related to supOU processes:

1) Log-NTS

2) Log-VG

3) Log-Euler's gamma

4) Log-z scenario

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