A continuous-time framework for ARMA processes

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Introduction

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We wish to consider a modeling framework for continuous-time stationary stochastic processes that has an ARMA-like structure. Let (L_t) be a Lévy process. Then this leads us to study stationary solutions to

$$Y_t - Y_s = \int_{\mathbb{R}} Y_u \, \phi_{s,t}(du) + \int_{\mathbb{R}} \theta_{s,t}(u) \, dL_u$$

where $\theta_{s,t} = \theta(t - \cdot) - \theta(s - \cdot)$ for a sufficiently regular function θ concentrated on $[0, \infty)$ and $\phi_{s,t} = \phi(t - \cdot) - \phi(s - \cdot)$ for a sufficiently regular signed measure ϕ concentrated on $[0, \infty)$.

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- if $\theta(u) = \mathbb{1}_{[0,\infty)}(u)$, $\theta_{s,t} = \mathbb{1}_{(s,t]}(u)$ and we get an increment in the Lévy process as noise.
- if $\theta(u) = u_+^\alpha$, $\alpha \in (-1/2, 1/2)$,

$$\theta_{s,t}(u) = (t-u)_+^\alpha - (s-u)_+^\alpha$$

and we get a fractional Lévy process as noise.

SDDE

If $\phi(du) = \eta((-\infty, u])du$ for a finite signed measure η, the equation may be rewritten as

$$Y_t - Y_s = \int_s^t \int_{[0,\infty)} Y_{u-v}\, \eta(dv)\, du + \int_{\mathbb{R}} \theta_{s,t}(u)\, dL_u.$$

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- measures η concentrated on a compact set.
- the Lévy driven case.

SDDE

We show existence and uniqueness of solutions to SDDEs when

- η does not necessarily have compact support which makes it possible to relate SDDEs and CARMA processes (more on this later).
- θ is such that the moving average integral exists which gives the possibility to introduce long-range dependence into the model.
- L_1 has first moment which is a more restrictive assumption than otherwise needed in the literature.

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The solution is given by

$$Y_t = \int_{\mathbb{R}} \theta * g(t-u) dL_u, \quad \mathcal{F}[g](y) = 1/(-iy - \mathcal{F}[\eta](y))$$

(under the standard assumption that $iy + \mathcal{F}[\eta](y) \neq 0$ for all $y \in \mathbb{R}$ and the mild assumption that η has second moment).

CARMA processes

A CARMA(p, q) process (Y_t) is stationary and satisfies the formal equation

$$P(D)Y_t = Q(D)DL_t, \quad t \in \mathbb{R},$$

where P, respectively Q are polynomials of order $p \in \mathbb{N}$, respectively $q \in \mathbb{N}_0$ with $p > q$. Here D denotes differentiation wrt. t.

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- CARMA$(1, 0)$ is an Ornstein-Uhlenbeck process.
- CARMA$(2, 1)$ is the stationary solution to

$$D^2 Y_t + a_1 DY_t + a_2 Y_t = b_0 DL_t + D^2 L_t.$$

Finding the solution

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$$\mathcal{F}[Y](y) = \frac{Q(-iy)}{P(-iy)}\mathcal{F}[DL](y) = \mathcal{F}[g \dot{*} DL](y)$$

where $g \in L^2$ is a function with Fourier transform $Q(-i\cdot)/P(-i\cdot)$ and $g \dot{*} DL(t) = \int_{\mathbb{R}} g(t-u)dL_u$. This agrees with the solution given in the literature.

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△

Inverting a CARMA process

We say that a CARMA process (Y_t) is invertible if $Q(z) \neq 0$ when $\mathrm{Re}(z) \geq 0$. Whenever this is the case,

$$\sum_{k=0}^{p-q-1} c_{k+1} d(D^k Y_t) = \int_{[0,\infty)} Y_{t-v} \eta(dv) dt + dL_t$$

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The representation gives

- a straightforward way to recover the noise when the process (Y_t) is observed.
- an intuitive dynamical representation of CARMA processes.

MSDDE

A multivariate SDDE is an equation on the form

$$dY_t = \int_{[0,\infty)} Y_{t-v}\eta(dv)dt + dZ_t, \quad t \in \mathbb{R},$$

where $(Y_t) \subseteq \mathbb{R}^{1\times n}$, η is a finite signed measure with second moment that take values in the space of $n \times n$ matrices, and $(Z_t) \subseteq \mathbb{R}^{1\times n}$ is a sufficiently regular stationary increment process.

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$$\det(iyI + \mathcal{F}[\eta](y)) \neq 0, \quad \text{for all } y \in \mathbb{R}.$$

The solution is given by

$$Y_t = Z * g(t), \quad \text{where} \quad \mathcal{F}[g](y) = (-iyI - \mathcal{F}[\eta](y))^{-1}.$$

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- higher order SDDEs, and therefore also invertible CARMA processes.
- invertible multivariate CARMA processes.

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- Let $\alpha \in (0, 1/2)$ and $(I^\alpha L_t)$ be a fractional Lévy process. Then a FICARMA process (Y_t) satisfy the formal equation

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- Given an estimate of α, calculate $(D^\alpha Y_t)$ and estimate the parameters in P and Q.
- Using the SDDE relation we may invert the CARMA relation and get the increments of $(I^\alpha L_t)$. Use this to estimate α and start over.

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Thank you for your attention!