Asymptotic behaviour of Gaussian minima

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jointly with Arijit Chakrabarty

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- $[a, b]$ a compact interval, $u > 0$ a high level.
- The scenario: the entire sample path of $\mathbf{X}$ on $[a, b]$ is above u.
- How do Gaussian minima behave when they are high?

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Question 2. Given the event

$$B_u := \left\{\min_{a\leq t\leq b} X_t > u\right\},$$

how does the conditional distribution of $(X_t : t \in [a, b])$ behave as $u \to \infty$?

Question 3. Conditionally on B_u, what is the overshoot

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Question 4. What is the asymptotic conditional distribution, given B_u, of the location of the minimum

$$\arg \min_{a \le t \le b} X_t \quad \text{as } u \to \infty?$$

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$$\lim_{u\to\infty} \frac{1}{u^2} \log P\left(\min_{a\le t\le b} X_t > u\right) = -\frac{1}{2\sigma_*^2(a,b)}\,.$$

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$$x(t) = \frac{1}{\sigma_*^2(a,b)} \int_a^b R(t,s)\,\nu_*(ds),\ \ a \le t \le b\,.$$

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- The canonical example: the Gaussian spectral density

$$F_X(dx) = e^{-x^2/2} \, dx, \; x \in \mathbb{R} \, .$$

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3 the optimal value $\sigma_*^2(b) > 0$.

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- Let Σ be the covariance matrix of $\big(X_{t_1}, \ldots, X_{t_k}\big)$.
- Let $\theta = (\theta_1, \ldots, \theta_k) = \Sigma^{-1}\mathbf{1}$.
- Then $\theta_j > 0$, $j = 1, \ldots, k$,

$$P\big(\min_{j=1,\ldots,k} X_{t_j} > u\big) \sim (2\pi)^{-k/2}(\det \Sigma)^{-1/2}(\theta_1 \ldots \theta_k)^{-1}$$

$$u^{-k} e^{-u^2/2\sigma_*^2(b)}, \; u \to \infty \,.$$

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- Then μ is infinitely differentiable, ≥ 1 on $[0, b]$.
- $\mu \equiv 1$ on S, so points of S are local minima.
- **The key assumption:** $\mu'' > 0$ on $S \cap (0, b)$.

Theorem 1

Let the cardinality of S be k. Then

$$P\left(\min_{0\le t\le b} X_t > u\right) \sim cu^{-k} e^{-u^2/2\sigma_*^2(b)},\ u\to\infty$$

for $c \in [0,\infty)$.

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for $c \in [0, \infty)$.

Furthermore, $c > 0$ if and only if the key assumption holds.

Theorem 2

Suppose the key assumption holds. Then in $C[0, b]$,

$$P\left((X_t - u\mu(t) : a \leq t \leq b) \in \cdot \,\middle|\, \min_{t\in[a,b]} X_t > u\right) \Rightarrow Q_W(\cdot)\,,$$

where Q_W is the law of a tilted Gaussian process on $[a, b]$.

Theorem 3

Suppose the key assumption holds.

Then, as $u \to \infty$, the conditional distribution of

$$u\Big(\min_{t\in[a,b]} X_t - u\Big) \quad \text{given} \quad \min_{t\in[0,b]} X_t > u$$

converges weakly to the exponential distribution with mean $\sigma_*^2(b)$.

Theorem 4

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Then, as $u \to \infty$,

$$P\left(T_* \in \cdot \,\Big|\, \min_{s\in[0,b]} X_s > u \right) \Rightarrow \nu_* \,.$$

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- **Example** The Gaussian covariance function $R(t) = e^{-t^2/2}$.
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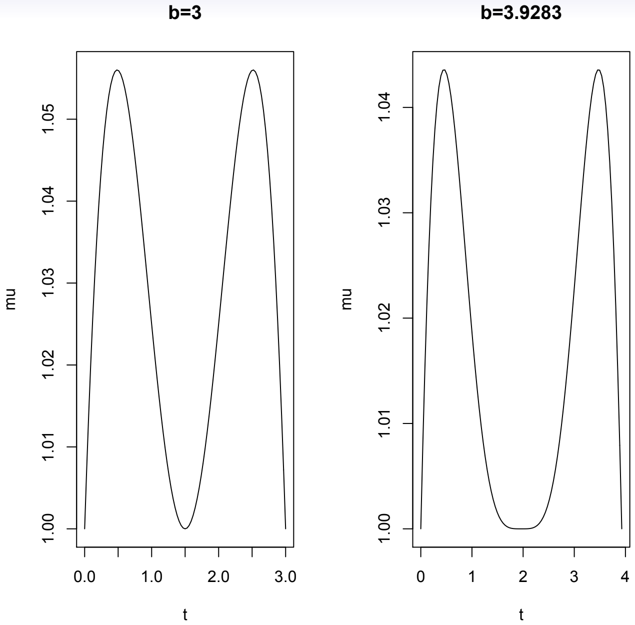
- Then $S = \{0, b/2, b\}$.


b=3
b=3.9283
mu
mu
1.00
1.01
1.02
1.03
1.04
1.05
1.00
1.01
1.02
1.03
1.04
0.0
1.0
2.0
3.0
0
1
2
3
4
t
t