Change of time and universal distributions in turbulence

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Abstract

The notion of universality is a key concept in the phenomenological description of turbulent flows. Here, we report on empirical evidence for a new type of universality that goes beyond the universal scaling picture of fully developed turbulent states. We show that the empirical densities of turbulent velocity increments obtained from widely different experiments, covering a wide range of Reynolds numbers, collapse after applying a deterministic time change in terms of the variances of velocity increments. Moreover, the conditional distributions of velocity increments also collapse in terms of the intrinsic time change after adjusting the conditional means.

It is of outstanding importance for a basic understanding of turbulence to extract universal features from the wide range of turbulent phenomena. Here universality refers to properties of turbulent flows that are, to a large extent, independent of flow conditions.

The large scale behaviour of turbulent flows strongly depends on the boundary conditions. Concentrating on small scales, including the inertial range and dissipation scales (below the inertial range), the notion of universality in turbulence goes back to Kolmogorov (1941). Kolmogorov hypothesized that, in the limit of very large Reynolds numbers, the statistics of dissipation scales only depend on the viscosity ν and the mean energy dissipation. In the inertial range, ν becomes irrelevant and the mean energy dissipation is the only relevant parameter.

With increasing quality and amount of experimental data, intermittency played a dominant role. As a consequence, the monofractal scaling of velocity increments according to Kolmogorov's 1941 theory has been extended to account for the universality of multifractal scaling exponents in the limit of large Reynolds numbers.

In terms of moments of temporal velocity increments

$$u_s(t) \equiv v_{s+t} - v_t, \; s > 0 \tag{1}$$

intermittency is usually described by (approximate) multifractal scaling of structure functions (e.g. Anselmet et al (1984) and Maurer et al (1994))

$$\mathrm{S}_n(s) = \mathrm{E}\{u_s(t)^n\} \propto s^{\tau(n)}. \tag{2}$$

Here, v_t is one component of the velocity (usually along the mean flow) at time t and at a fixed position and the time scale s is within the inertial range. $\mathrm{E}\{\ \}$ denotes the expectation. The exponents $\tau(n)$ are called the scaling exponents and show a non-linear dependence on the order n which is expected to be universal in the large Reynolds number limit. In the present discussion we restrict to stationary turbulent flows and consequently, we do not refer to the time t on the left hand side of (1) when dealing with stationary expectations.

Multifractal scaling of structure functions is assumed to hold in the limit of infinite Reynolds number (Frisch (1995)). However, experiments show that the scaling behaviour (2) might be poor, even for large Reynolds numbers (Sreenivasan and Antonia (1997) and Arneodo et al (1996)). Furthermore, even if the scaling relation (2) holds, the inertial range still covers only part of the accessible scales where intermittency is observed.

From a probabilistic point of view, (2) expresses a scaling relation for the moments of the probability density function (pdf) of velocity increments. A proper estimation of higher-order moments requires an accurate estimation of the tails of the pdf. Thus it may be advantageous to directly work with the pdf. In terms of the pdf, intermittency refers to the increase of the non-Gaussian behaviour of the pdf of velocity increments with decreasing time scale.

A typical scenario is characterized by an approximate Gaussian shape for the large scales (larger than scales at the inertial range), turning to exponential tails within the inertial range and stretched exponential tails for dissipation scales (below the inertial range). This change of shape across all scales clearly reveals the inadequacy of a characterization of intermittency solely via multifractal scaling of structure functions (which is observed only within the inertial range).

Existing work about the pdf of velocity increments concentrates on a description of the tails of the pdf or characterizes the pdf only within the inertial range (Castaing et al (1990), Gagne et al (1994), Chabaud et al (1994), Praskovsky and Oncley (1994), Vincent and Meneguzzi (1991), Kailasnath et al (1992), Stolovitzky et al (1993), Tabeling et al (1996), Lewis and Swinney (1999), Noullez et al (1997), van de Water and Herweijer (1999), Benzi et al (1991), Tchéou et al (1999), Arimitsu and Arimitsu (2001), Beck (2000), Beck et al (2001) and Renner et al (2001)).

Different turbulent data show an individual evolution of the densities of velocity increments across time scales. Here, we report on universal features of these evolutions across time scales. Recent studies of various high quality data sets revealed a hidden type of universality that can be expressed as a stochastic equivalence class (SEC) of the form (Barndorff-Nielsen et al (2004), Barndorff-Nielsen and Schmiegel (2006) and Barndorff-Nielsen et al (2006))

$$\frac{u_{s_1}^{(i)}}{\sqrt{\mathrm{Var}(v^{(i)})}} \stackrel{d}{=} \frac{u_{s_2}^{(j)}}{\sqrt{\mathrm{Var}(v^{(j)})}} \;\Leftrightarrow\; c_2^{(i)}(s_1) = c_2^{(j)}(s_2) \tag{3}$$

where $\stackrel{d}{=}$ denotes equality in distribution and $c_2(s) = \mathrm{Var}(u_s/\sqrt{\mathrm{Var}(v)})$ denotes the variance of normalized velocity increments at time scale s. The superscripts (i) and (j) label different experiments. Relation (3) states that the densities of velocity increments follow a one-parameter curve in the space of probability distributions. Each individual data set covers a certain part of this one-parameter curve. It is the variance $c_2(s)$ that accounts for the individual characteristics of each data set and determines the location onto this one-parameter curve.

The data sets analyzed with respect to the SEC relation (3) are time series of the velocity $v(t)$ (in direction of the mean flow) from helium jet experiments (data sets $(h85)$, $(h89)$, $(h124)$, $(h209)$, $(h283)$, $(h352)$, $(h463)$, $(h703)$, $(h885)$, $(h929)$ and $(h1181)$), from an atmospheric boundary layer experiment (data set (at)), from a wind tunnel experiment (data set (w)) and from a free air jet experiment (data set (air)). Each data set is normalized by its standard deviation. Table 1 lists the data sets together with the sampling frequency f and the Taylor scale based Reynolds number. For more details about the data sets we refer to Chanal et al (2000) for the helium jet experiments, to Dhruva (2000) and Sreenivasan and Dhruva (1998) for the atmospheric boundary layer experiment, to Antonia and Pearson (2000) for the wind tunnel experiment and to Renner et al (2001) for the free air jet experiment.

Figure 1 shows the collapse of the densities of velocity increments for the different flow experiments at time scales s (in units of the finest resolution $1/f$) where the variances $c_2(s)$ are the same. After applying a time change from time scale s to the variance $c_2(s)$ the densities of velocity increments from widely different data sets, different in Reynolds number and different in experimental set-up, collapse onto universal distributions, independent of the flow conditions (Barndorff-Nielsen and Schmiegel (2006)).

Since the variance $c_2(s)$ is monotonically increasing for turbulent velocity increments, we can express the relation (3) in terms of an intrinsic, deterministic time scale change. Taking any two experiments (i) and (j) and some time scale $s^{(j)}$ for experiment (j), the corresponding time scale $s^{(i)}$ in experiment (i) (at which the corresponding densities of velocity increments collapse) follows the one-to-one time scale change

$$s^{(i)} = \psi^{(i,j)}(s^{(j)}) = \overline{c}_2^{(i)}(c_2^{(j)}(s^{(j)})), \tag{4}$$

where $\overline{c}_2$ denotes the inverse of c_2.

Figure 2 shows the intrinsic time change $\psi^{(i,j)}$ for some of the data sets. For a wide range of time scales, the intrinsic time change is linear in first approximation. In case of a purely linear time change the SEC relation (3) would follow from the corresponding relation for the velocities itself

$$\frac{v^{(i)}_{\psi^{(i,j)}(s)}}{\sqrt{\mathrm{Var}(v^{(i)})}} \stackrel{law}{=} \frac{v^{(j)}_s}{\sqrt{\mathrm{Var}(v^{(j)})}} \tag{5}$$

where $\stackrel{law}{=}$ denotes equality in law. For the relation (5) to hold true, the linearity of $\psi^{(i,j)}$ is essential, since only a linear time change preserves stationarity.

The apparent non-linearity of $\psi^{(i,j)}$ at very small and very large time scales implies that the SEC relation (3) can not be generalized to the corresponding stochastic relation (5) between the pure velocity records.

The SEC relation in the form of (3) states a distributional relation between velocity increments obtained in different experiments. A natural question addresses possible dynamical aspects of this relation. The strongest dynamical interpretation in the form of (5) fails due to the non-linearity of ψ. A milder dynamical interpretation of (3) concerns the conditional distributions of velocity increments and their relation to the SEC relation (3). Figure 3 (left column) shows the conditional densities

$$p(u_s(t)|u_s(t-s)=c) = p(u_s(t)|c - c_{\min} \leq u_s(t-s) \leq c + c_{\max}) \tag{6}$$

for data sets (i) ($h89$) and (j) ($h929$) and for various values of c. The displayed time lags are $s^{(j)} = 13$ and $s^{(i)} = \psi^{(i,j)}(s^{(j)}) = 41$ (in units of the corresponding finest resolutions $1/f$). The conditioning intervals are bounded by $c_{\min} = c_{\max} = (\max\{u_s^{(i)}(t)\} - \min\{u_s^{(i)}(t)\})/12$ (which gives a reasonable number of observations). The application of the intrinsic time change ψ does not cause the conditional densities to collapse for all conditioning values c. However, the shape of the conditional densities for experiment (i) and experiment (j) is the same for each conditioning value c. This shape equivalence is shown in Figure 3 (right column) displaying the same densities after adjusting the conditional means, i.e. the right column shows the conditional densities $p(u^{(j)}_{s^{(j)}}(t)|u^{(j)}_{s^{(j)}}(t-s^{(j)})=c)$ and $p(u^{(i)}_{s^{(i)}}(t)-b^{(i,j)}(s^{(j)})|u^{(i)}_{s^{(i)}}(t-s^{(i)})=c)$ where

$$\begin{aligned} b^{(i,j)}(s^{(j)}) =& \mathrm{E}\left\{u^{(i)}_{s^{(i)}}(t)|c-c_{\min}\leq u^{(i)}_{s^{(i)}}(t-s^{(i)})\leq c+c_{\max}\right\} \\ &-\mathrm{E}\left\{u^{(j)}_{s^{(j)}}(t)|c-c_{\min}\leq u^{(j)}_{s^{(j)}}(t-s^{(j)})\leq c+c_{\max}\right\} \end{aligned} \tag{7}$$

is the difference of the conditional means. The same shape invariance and collapse of the conditional densities after adjusting the conditional means is also observed for the other data sets listed in Table 1.

Besides a shift in terms of the conditional means, the conditional densities of velocity increments collapse after applying the intrinsic time change ψ. These results suggest that the bivariate distributions of time wise turbulent velocity increments are universal in terms of the intrinsic time change ψ and after adjusting the conditional means. It is the variance of velocity increments as a function of the time scale and the conditional means as a function of the time scale that contain the individual characteristics of the bivariate distributions of velocity increments obtained in the different turbulent experiments. In other words, the bivariate distributions of velocity increments only depends on two functions. For the marginal distributions of velocity increments we are left with one function, the variance $c_2(s)$ as a function of the time scale s. These individual characteristics, $c_2(s)$ and the conditional means, are different for each experiment. The universality arises as a universal dependence of the marginal distributions on $c_2(s)$ and the universal dependence of the bivariate distributions on $c_2(s)$ and the conditional means.

This parsimonious description of turbulent velocity increments has some immediate consequences for modelling and simulation of turbulent flows, as well as for turbulence theory. In particular, the unique dependence of the marginal distributions of velocity increments on the variance implies that structure fuctions of order n depend on the structure function of order 2 in a universal way. This universal dependence opens the door for a solution of the long standing closure problem in turbulence, if this universal dependence is known. Promising advances in this direction are reported in Barndorff-Nielsen et al (2004) where it was shown that the marginal distributions of velocity increments are well described by normal inverse Gaussian distributions. These distributions depend on 3 parameters (for stationary flows) with a universal dependence on the variance of velocity increments, in accordance with the empirical findings reported here.

The origin of the existence of SEC in turbulence is theoretically motivated in Barndorff-Nielsen et al (2004) and Barndorff-Nielsen and Schmiegel (2006). There it was shown that SEC can be derived as a consequence of Extended Self-Similarity (Benzi et al (19993)) and the scaling limit of fully developed turbulence. On the other hand, it is shown in Barndorff-Nielsen and Schmiegel (2006) that the combined assumption of SEC and the existence of a scaling limit in fully developed turbulence implies Extended Self-Similarity. These findings show that the existence of a scaling limit in fully developed turbulence is the key mechanism that links SEC and Extended Self-Similarity.

The concept of SEC is not restricted to turbulent flows. A similar type of SEC has also been found for financial markets (Barndorff-Nielsen et al (2006)). These findings add a new type of similarity between the statistics of turbulent flows and financial markets (other similarities are discussed in Ghashgaie et al (1996), Peinke et al (2004) and Barndorff-Nielsen (1998)).

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data set	(at)	(w)	(air)	(h85)	(h89)	(h124)	(h209)
R_λ	17000	80	190	85	89	124	209
f	5000	10000	8000	5742	2871	2296	4593
data set	(h283)	(h352)	(h463)	(h703)	(h885)	(h929)	(h1181)
R_λ	283	352	463	703	885	929	1181
f	1523	2296	9187	2715	3675	3675	3675

Table 1: The sampling frequencies f and the Taylor scale based Reynolds numbers R_λ for the helium jet experiments (data sets ($h85$), ($h89$), ($h124$), ($h209$), ($h283$), ($h352$), ($h463$), ($h703$), ($h885$), ($h929$) and ($h1181$), kindly provided by B. Chabaud), the atmospheric boundary layer experiment (data set (at), kindly provided by K.R. Sreenivasan), the wind tunnel experiment (data set (w), kindly provided by B.R. Pearson) and the free air jet experiment (data set (air), kindly provided by J. Peinke).

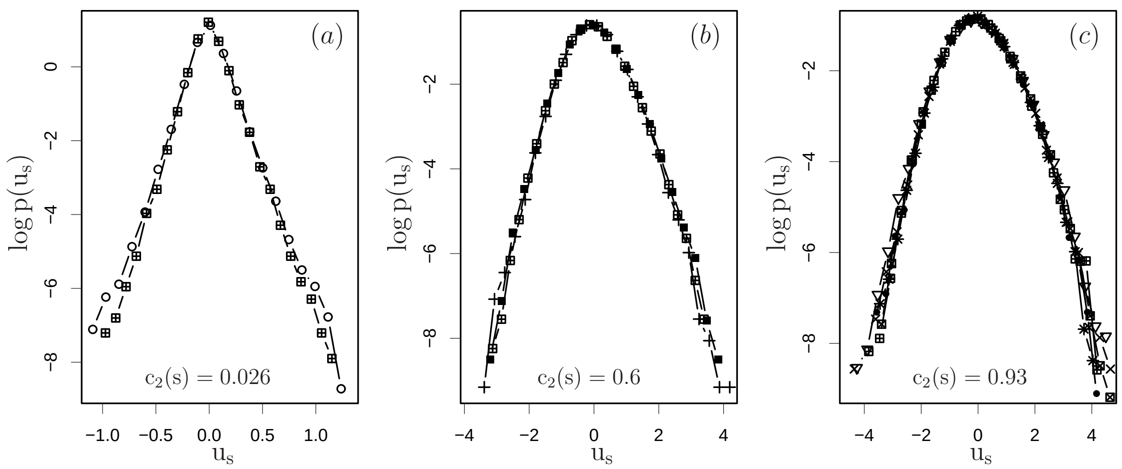

Figure 1: Collapse of the densities $p(u_s)$ of velocity increments for various fixed values of the variance $c_2(s)$. The corresponding values of the time lag s (in units of the finest resolution $1/f$ of the corresponding data sets) and the data sets are (a) ($s = 116, \mathrm{at}$) ($\circ$), ($s = 4, \mathrm{h352}$) ($\boxplus$), (b) ($s = 192, \mathrm{h885}$) ($\blacksquare$), ($s = 88, \mathrm{h352}$) ($\boxplus$), ($s = 10, \mathrm{w}$) ($+$) and (c) ($s = 420, \mathrm{h703}$) ($\times$), ($s = 440, \mathrm{h929}$) ($\triangledown$), ($s = 180, \mathrm{h352}$) ($\boxplus$), ($s = 270, \mathrm{h283}$) ($\bullet$), ($s = 108, \mathrm{h124}$) ($*$), ($s = 56, \mathrm{h85}$) ($\boxtimes$).

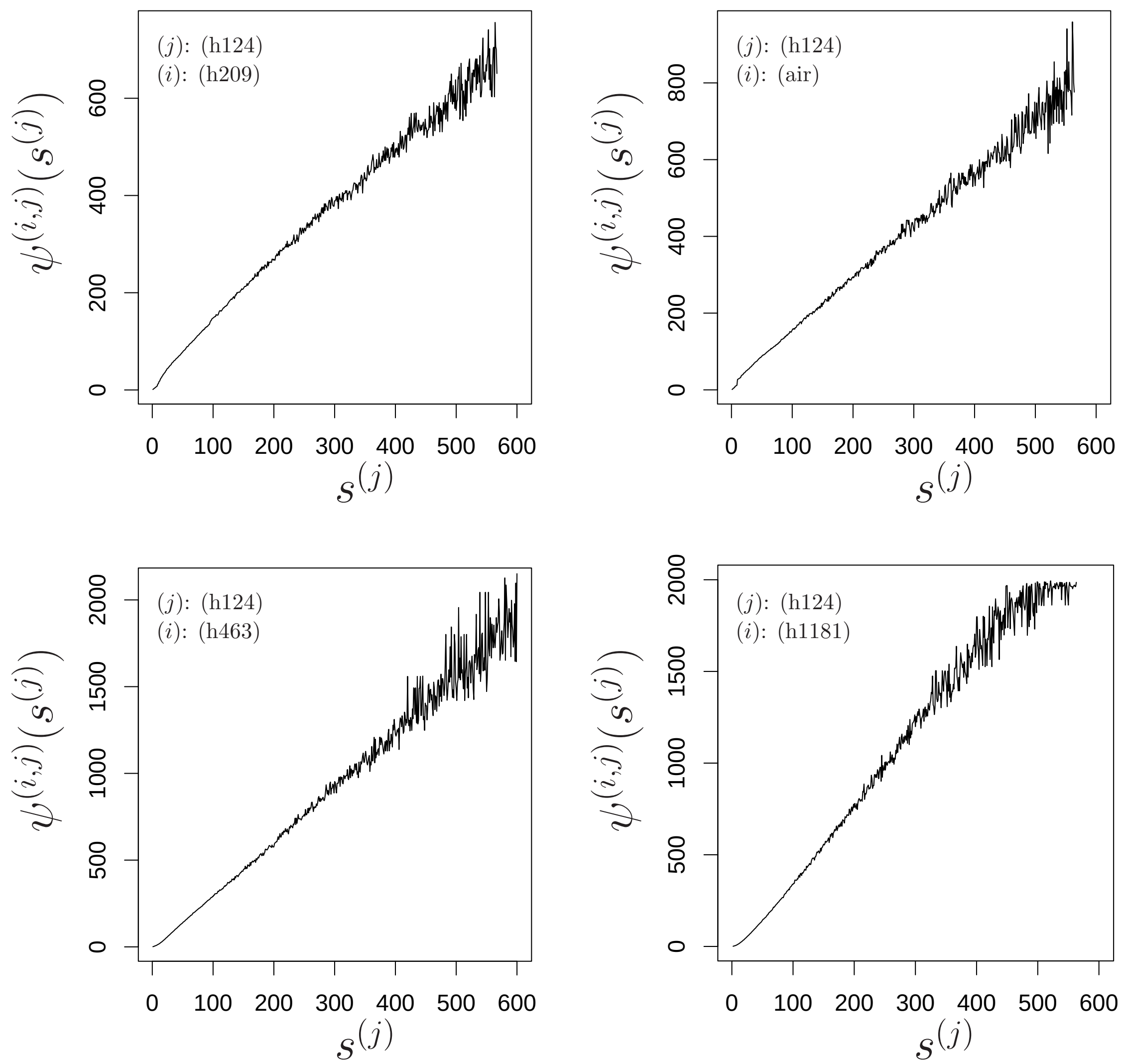

Figure 2: The intrinsic time change $\psi^{(i,j)}(s^{(j)})$ as a function of $s^{(j)}$ (in units of the finest resolution of the corresponding data sets) for various turbulent data sets.

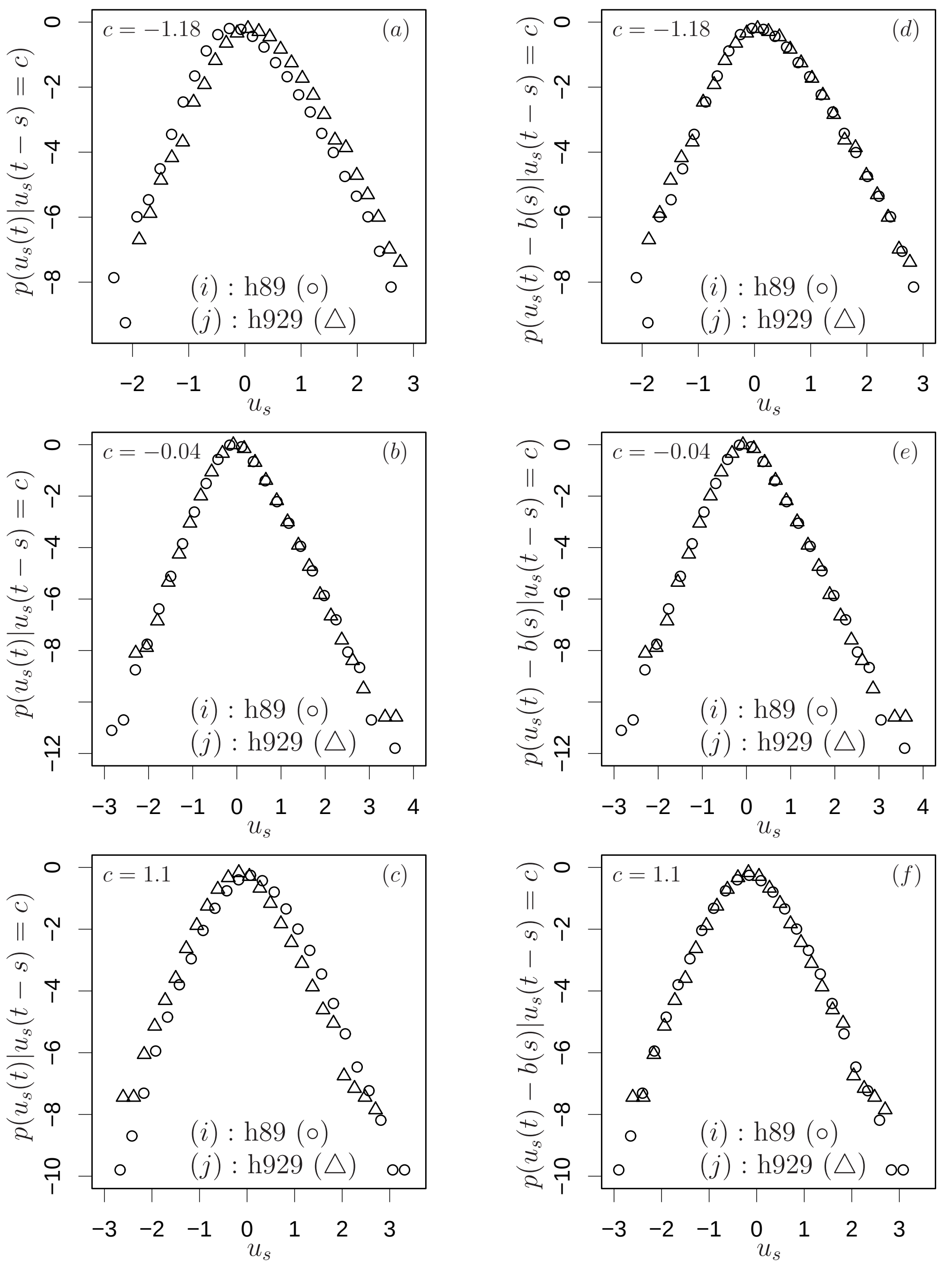

Figure 3: The logarithm of the conditional densities $p(u^{(j)}_{s^{(j)}}(t)|u^{(j)}_{s^{(j)}}(t-s^{(j)})=c)$ and $p(u^{(i)}_{s^{(i)}}(t)|u^{(i)}_{s^{(i)}}(t-s^{(i)})=c)$ (left column) for data sets (i) *h*89 (○) and (j) *h*929 (△) and (a) $c=-1.18$, (b) $c=-0.04$ and (c) $c=1.1$. The right column shows the logarithm of the conditional densities $p(u^{(j)}_{s^{(j)}}(t)|u^{(j)}_{s^{(j)}}(t-s^{(j)})=c)$ and $p(u^{(i)}_{s^{(i)}}(t)-b^{(i,j)}(s^{(j)})|u^{(i)}_{s^{(i)}}(t-s^{(i)})=c)$ for the same data sets and (d) $c=-1.18$, (e) $c=-0.04$ and (f) $c=1.1$. The corresponding time scales are $s^{(j)}=41$ and $s^{(i)}=\psi^{(i,j)}(s^{(j)})=13$ (in units of the finest resolution $1/f$ of the corresponding data sets).