Lévy based growth models

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Abstract

In the present paper, we propose a class of continuous time spatio-temporal models based on Lévy theory for growing planar objects. The focus is on modelling of the radial function $R_t(\phi)$ at time t and angle ϕ of planar star-shaped objects. We study Lévy based models for the time derivative of the radial function

$$\frac{\partial}{\partial t}R_t(\phi) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi;\phi)Z(d\xi),\ \phi \in [-\pi,\pi),$$

where Z is a Lévy basis, $A_t(\phi) \subseteq [-\pi,\pi) \times \mathbb{R}$ is a so-called ambit set, $f_t(\cdot;\phi)$ is a deterministic weight function and μ_t a deterministic function. The induced model for the radial function is of the same type. We also consider similar models for transformations of the radial function. An important advantage of these models is that explicit expressions for

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$$

can be derived in terms of the components of the model. As a bi-product, our modelling approach provides new flexible models for space-time covariance fucntions on the circle. We also show that a particular version of our growth model can be regarded as a continuous analogue of the famouns Richardson growth model. The flexibility of the appraoch is exemplified by simulation of a variety of growth patterns. An application of the Lévy based growth models to tumour growth is discussed.

KEYWORDS: growth models, Lévy basis, spatio-temporal modelling, tumour growth.

1 Introduction

Stochastic spatio-temporal modelling is of great importance in a variety of disciplines of natural science, including biology (?, ?, ?, ?), image analysis (?), geophysics (?, ?, ?) and turbulence (?), just to name a few. In particular, modelling of tumour growth dynamics has been a very active research area in recent years (?, ?, ?). In most of the above cited works, the model is given implicitly and the resulting dynamics are difficult to control explicitly. However, for applications and for the theoretical understanding of the employed modelling framework it is essential to connect the ingredients of the model with dynamical and spatial properties of the system in focus. Furthermore, for a parsimonious description of systems, different with respect to the dynamics and physical mechanisms underlying the dynamics, it is desirable to have at hand a flexible and, at the same time, mathematically tractable modelling framework.

Lévy based models are a promising modeling framework to meet these requirements concerning flexibility and dynamical control. Until now, Lévy based models have proven useful for describing such widely different systems as turbulent flows (**?**, **?**, **?**) and spatio-temporal Cox processes (**?**,**?**) The purpose of the present paper is to study growth modelling in a Lévy based framework, i.e. stochastic spatio-temporal modelling based on the integration with respect to a Lévy basis (an infinitely divisible and independently scattered random measure). The paper is a natural continuation of the work initiated in **?** which was mainly directed towards an audience of physicists.

In the growth literature, there is a variety of growth models for objects in discrete space, cf. e.g. **?**, **?**, **?**, **?** and references therein. An important early example is the Richardson model, introduced in **?**. Here, the growth is described by a Markov process. For a growing object in the plane, the state at time t is a random subset Y_t of $\mathbb{Z}^2$ consisting of the 'infected sites'. An uninfected site is transferred to an infected site with a rate proportional to the number of infected nearest neighbours. It can be shown that if Y_0 consists of a single site, Y_t/t has a non random shape as $t \to \infty$.

More recently, a growth model in continuous space has been discussed in **?**. For planar objects, the model is constructed from a spatio-temporal Poisson point process on $\mathbb{R}^3$

$$Z = \{(x_i, t_i)\}.$$

The random growing object $Y_t \subset \mathbb{R}^2$ is a subset of

$$\cup_{\{i: t_i \leq t\}} B(x_i, r),$$

constructed such that Y_t is always connected. Here, $B(x, r)$ is a circular disc with centre x and radius r. In this model t_i is thought of as a time point of outburst and x_i is the location of the outburst in the tumour, say. A closely related discrete time Markov growth model has been discussed in detail in **?**. This model can be characterised as a sequence of Boolean models

$$Y_{t+1} = \cup\{B(x_i, r) : x_i \in Y_t\},$$

where $\{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^2$.

In the present paper, we study instead how such a point process can be used to describe the growing boundary of the object $Y_t \subseteq \mathbb{R}^2$. We consider a Poisson point process $Z = \{(\theta_i, t_i)\}$ on $(-\pi, \pi] \times \mathbb{R}$ and let the rate of growth at time t in direction ϕ be

$$\frac{\partial}{\partial t} R_t(\phi) = \sum_{\xi \in Z \cap A_t(\phi)} f_t(\xi; \phi), \tag{1}$$

where

$$A_t(\phi) \subset \{(\theta, s) : s \leq t\}$$

is a subset of the past of time t, a so-called ambit set[1], and $f_t(\cdot; \phi)$ is a non-negative weight function. We will always assume that $(\phi, t) \in A_t(\phi)$. See Figure **??** for an illustration.

It turns out that the model (**??**) is closely related to the Richardson model. To see this, let us consider for each $\phi \in (-\pi, \pi]$ the stochastic time transformation $t \to R_t(\phi)$. We can then represent the ambit set $A_t(\phi)$ as a subset of Y_t

$$\tilde{A}_t(\phi) = \{(R_s(\theta) \cos\theta, R_s(\theta) \sin\theta) : (\theta, s) \in A_t(\phi)\}$$

[1]Latin: *ambitus*. 1. The bounds or limits of a place or district. 2. A sphere of action, expression or influence.

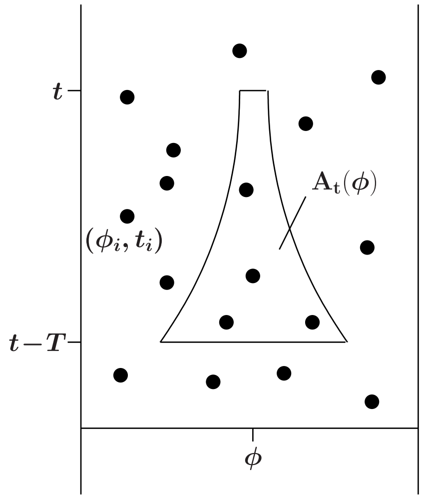

Figure 1: Illustration of the growth model (**??**). The growth rate at time t in direction ϕ depends on the number of points from the point process $Z = \{(\theta_i, t_i)\}$ falling in the ambit set $A_t(\phi)$.

that touches the boundary of Y_t at the point $(R_t(\phi)\cos\phi, R_t(\phi)\sin\phi)$. Likewise we can represent the part of the spatio-temporal point process Z, arrived before time t,

$$Z_t = \{(\theta_i, t_i) : t_i \leq t\},$$

as a subset of Y_t

$$\tilde{Z}_t = \{(R_{t_i}(\theta_i)\cos\theta_i, R_{t_i}(\theta_i)\sin\theta_i) : t_i \leq t\}.$$

We can think of $\tilde{Z}_t$ as locations of outbursts at time points before t. Finally, if we let

$$\tilde{f}_t((s\cos\theta, s\sin\theta); \phi) = f_t((\theta, s); \phi),$$

the fundamental equation (**??**) can be written as

$$\frac{\partial}{\partial t} R_t(\phi) = \sum_{\tilde{\xi} \in \tilde{Z}_t \cap \tilde{A}_t(\phi)} \tilde{f}_t(\tilde{\xi}; \phi). \tag{2}$$

According to (**??**), the growth rate at time t in direction ϕ is determined by the outbursts at time points before t, lying in the stochastic neighbourhood $\tilde{A}_t(\phi)$. Figure **??** illustrates the set $\tilde{A}_t(\phi)$. In this fashion our model may be regarded as a continuous analogue of the Richardson model.

The model (**??**) is just one example of the models to be considered in the present paper. More generally, we consider models of the type

$$\frac{\partial}{\partial t} R_t(\phi) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) Z(d\xi). \tag{3}$$

Here, μ_t is a deterministic function, contributing to the overall growth pattern, while the stochastic integral determines the dependence structure in the growth process via the ambit set $A_t(\phi)$ and the Lévy basis Z. The submodel class with $\mu_t \equiv 0$ and Z a Poisson basis has been presented above.

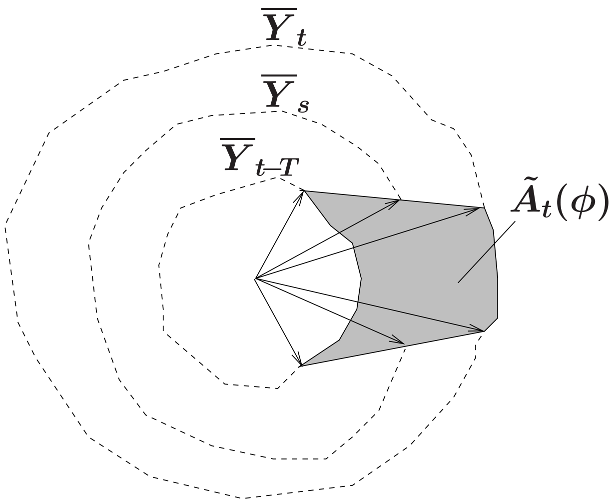

Figure 2: Illustration of the set $\tilde{A}_t(\phi)$ as the union of the boundaries of the objects Y_s within the angular bandwith $[\phi - \Theta(s), \phi + \Theta(s)]$ for $t - T \leq s \leq t$. As an example three profiles at times t, $t - T$ and $s \in [t - T, t]$ are shown. The relevant parts of the profiles, contributing to $\tilde{A}_t(\phi)$, are shown bold.

The organisation of the paper is as follows. Section 2 provides some background on Lévy bases which is the essential component in our approach. In Section 3, Lévy based growth models are studied while Section 4 contains explicit results for the covariance functions. In Section 5, an application of the Lévy based growth models to tumour growth is discussed while problems for future resarch is collected in Section 6. Section 7 concludes the paper. Basic results on moments of stochastic integrals with respect to a Lévy basis is summarized in Appendix A while a specific subclass of spatio-temporal covariance functions is studied in Appendix B.

2 Background

This section provides a brief overview of the theory of Lévy bases, in particular, the theory of integration with respect to a Lévy basis.

We focus on results which are prerequisites for subsequent sections, without proofs. For a more detailed study of infinitely divisible and independently scattered random measures and their theory of integration, see ?, ?, ? and ?. We will use the following notation $C(\lambda \ddagger X) = \log \mathbb{E}(e^{i\lambda X})$ for the cumulant function of a random variable X and $K(\lambda \ddagger X) = \log \mathbb{E}(e^{\lambda X})$ for the logarithm of the Laplace transform of X. The latter function will be called the kumulant function.

Let $(\mathcal{R}, \mathcal{B})$ be a measurable space. For concreteness we shall assume that $\mathcal{R} = \mathcal{S} \times \mathbb{R}$ where $\mathcal{S}$ is a Borel subset of $\mathbb{R}^d$. Furthermore, $\mathcal{B}$ is the Borel $\sigma-$algebra of $\mathcal{R}$. A collection of random variables $Z = \{Z(A) : A \in \mathcal{B}\}$ on $(\mathcal{R}, \mathcal{B})$ is said to be an *independently scattered random measure*, if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{B}$, the random variables $Z(A_n)$ are independent and $Z(\bigcup A_n) = \sum Z(A_n)$ a.s. Moreover, if $Z(A)$ is *infinitely divisible* for all $A \in \mathcal{B}$, Z is called a *Lévy basis*, cf. ?.

When Z is a Lévy basis, the cumulant function of $Z(A)$ can, by the famous Lévy-

Khintchine representation, be written as

$$C(\lambda \ddagger Z(A)) = i\lambda a(A) - \frac{1}{2}\lambda^2 b(A) + \int_{\mathbb{R}} (e^{i\lambda u} - 1 - i\lambda u \mathbf{1}_{[-1,1]}(u)) U(du, A), \tag{4}$$

$A \in \mathcal{B}$, where a is a signed measure on $\mathcal{B}$, b is a positive measure on $\mathcal{B}$, $U(du, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{B}$ and a measure on $\mathcal{B}$ for fixed du. The Lévy basis Z is said to have characteristics (a, b, U). The measure U will be referred to as the generalised Lévy measure.

Without loss of generality (for details, see **?**) we can assume that there exists a measure μ on $\mathcal{B}$ such that the generalised Lévy measure factorises as

$$U(du, d\xi) = V(du, \xi)\mu(d\xi),$$

where $V(du, \xi)$ is a Lévy measure for fixed ξ. Furthermore, the measures a and b are absolutely continuous with respect to the measure μ, i.e.

$$a(d\xi) = \tilde{a}(\xi)\mu(d\xi), \quad b(d\xi) = \tilde{b}(\xi)\mu(d\xi).$$

Under these assumptions we may think of

$$i\lambda\tilde{a}(\xi) - \frac{1}{2}\lambda^2\tilde{b}(\xi) + \int_{\mathbb{R}} \{e^{i\lambda u} - 1 - i\lambda u \mathbf{1}_{[-1,1]}(u)\} V(du, \xi)$$

as a cumulant function of a random variable $Z'(\xi)$ satisfying

$$C(\lambda \ddagger Z(d\xi)) = C(\lambda \ddagger Z'(\xi))\mu(d\xi). \tag{5}$$

If $\tilde{a}(\xi)$, $\tilde{b}(\xi)$ and the Lévy measure $V(\cdot; \xi)$ do not depend on ξ, we call Z a *factorisable* Lévy basis and then $Z'(\xi) = Z'$ does also not depend on ξ. If, moreover, μ is proportional to the Lebesgue measure, then Z is called a *homogeneous* Lévy basis and all finite dimensional distributions of Z are translation invariant.

Let us now consider the integral of a measurable function f on $\mathcal{R}$ with respect to a Lévy basis Z. For simplicity we denote this integral by $f \bullet Z$. Important for many calculations are the following equation for the cumulant function of the stochastic integral $f \bullet Z$ (subject to minor regularity conditions, cf. **?**)

$$C(\lambda \ddagger f \bullet Z) = \int C(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi). \tag{6}$$

The result (**??**) can heuristically be derived from (**??**). A similar result holds for the logarithm of the Laplace transform of $f \bullet Z$ (assumed to be finite),

$$K(\lambda \ddagger f \bullet Z) = \int K(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi). \tag{7}$$

We will now give a few examples of Lévy bases.

Example 1. (Gaussian Lévy basis) If Z is a Lévy basis with $Z(A) \sim N(a(A), b(A))$, where a is a signed measure on $\mathcal{B}$ and b is a positive measure on $\mathcal{B}$, we call Z a Gaussian Lévy basis. The Gaussian Lévy basis has characteristics $(a, b, 0)$ and the cumulant function is

$$C(\lambda \ddagger Z(A)) = i\lambda a(A) - \frac{1}{2}\lambda^2 b(A).$$

We have that $Z'(\xi) \sim N(\tilde{a}(\xi), \tilde{b}(\xi))$, i.e. $C(\lambda \ddagger Z'(\xi)) = i\lambda\tilde{a}(\xi) - \frac{1}{2}\lambda^2\tilde{b}(\xi)$. Furthermore,

$$C(\lambda \ddagger f \bullet Z) = \int C(\lambda f(\xi) \ddagger Z'(\xi))\mu(d\xi) = i\lambda(f \bullet a) - \frac{1}{2}\lambda^2(f^2 \bullet b). \quad (8)$$

Note that $f \bullet Z \sim N(f \bullet a, f^2 \bullet b)$. □

Example 2. (Lévy jump basis) A Lévy basis is called a Lévy jump basis if the characteristics of the basis is $(a, 0, U)$. In Table **??** we specify the functions V and $\tilde{a}$ for three important examples of Lévy jump bases, the Poisson basis, the Gamma basis and the inverse Gaussian basis. We also list the distribution of the random variable $Z'(\xi)$, its cumulant function, mean and variance.

	Poisson	Gamma	Inverse Gaussian
$V(du, \xi)$	$\delta_1(du)$	$\mathbf{1}_{\mathbb{R}_+}(u)\beta u^{-1}e^{-\alpha(\xi)u}du$	$\frac{\eta}{\sqrt{2\pi}}\mathbf{1}_{\mathbb{R}_+}(u)u^{-\frac{3}{2}}e^{-\frac{1}{2}\gamma^2(\xi)u}du$
$\tilde{a}(\xi)$	1	$\beta\left(\frac{1-e^{-\alpha(\xi)}}{\alpha(\xi)}\right)$	$\frac{\eta}{\sqrt{2\pi}}\int_0^1 u^{-\frac{1}{2}}e^{-\frac{1}{2}\gamma^2(\xi)u}du$
$Z'(\xi)$	Po(1)	$\Gamma(\beta, \alpha(\xi))$	$\mathrm{IG}(\eta, \gamma(\xi))$
$C(\lambda \ddagger Z'(\xi))$	$e^{i\lambda} - 1$	$-\beta\log\left(1 - \frac{i\lambda}{\alpha(\xi)}\right)$	$\eta\gamma(\xi)\left(1 - \sqrt{1 - \frac{2i\lambda}{\gamma^2(\xi)}}\right)$
$\mathbb{E}(Z'(\xi))$	1	$\frac{\beta}{\alpha(\xi)}$	$\frac{\eta}{\gamma(\xi)}$
$\mathbb{V}(Z'(\xi))$	1	$\frac{\beta}{\alpha^2(\xi)}$	$\frac{\eta}{\gamma^3(\xi)}$

Table 1: The definition of three Lévy jump bases, the Poisson basis, the Gamma basis and the inverse Gaussian basis, and the distribution of $Z'(\xi)$, with the corresponding cumulant function, mean and variance.

Note that if Z is a Poisson basis, $Z(A) \sim \mathrm{Po}(\mu(\mathrm{A}))$ with probability function

$$\frac{e^{-\mu(A)}\mu(A)^x}{x!}, \; x = 0, 1, 2, \ldots,$$

for all $A \in \mathcal{B}$. If $\alpha(\xi) \equiv \alpha$, $\gamma(\xi) \equiv \gamma$, then we have for all $A \in \mathcal{B}$, that $Z(A) \sim \Gamma(\beta\mu(A), \alpha)$ with density

$$\frac{\alpha^{\beta\mu(A)}}{\Gamma(\beta\mu(A))}x^{\beta\mu(A)-1}e^{-\alpha x}, \quad x > 0,$$

and $Z(A) \sim \mathrm{IG}(\eta\mu(\mathrm{A}), \gamma)$ with density

$$\frac{\eta\mu(A)e^{\eta\mu(A)\gamma}}{\sqrt{2\pi}}x^{-3/2}\exp\left\{-\frac{1}{2}\left((\eta\mu(A))^2 x^{-1} + \gamma^2 x\right)\right\}, \quad x > 0,$$

respectively. □

It follows from (**??**) that any Lévy basis Z can be expressed as the sum of a Lévy jump basis Z_1 and an independent zero mean Gaussian basis Z_2.

3 Lévy based growth models

In this section we study how the Lévy setup can be used to construct flexible stochastic models for growing objects. We focus on planar objects but generalisations to higher dimensions are straightforward. We denote the planar object at time t by $Y_t \subset \mathbb{R}^2$ and we will assume that Y_t is compact and star-shaped with respect to a point $z \in \mathbb{R}^2$ for all t. The boundary of the star-shaped object Y_t can be determined by its radial function $R_t = \{R_t(\phi) : \phi \in [-\pi, \pi)\}$, where

$$R_t(\phi) = \max\{r : z + r(\cos\phi, \sin\phi) \in Y_t\}, \ \ \phi \in [-\pi, \pi),$$

cf. Figure **??**.

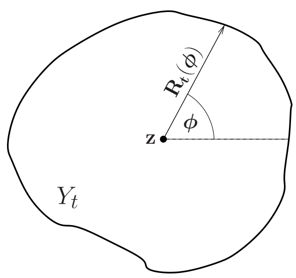

Figure 3: The star-shaped object Y_t is determined by its radial function $R_t(\phi)$ at time t and angle ϕ.

The growth rate will be described by the equation

$$\frac{\partial}{\partial t} R_t(\phi) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) Z(d\xi). \tag{9}$$

Here, the deterministic function μ_t contributes to the overall growth pattern while the stochastic integral determines the dependence structure in the growth process. The set $A_t(\phi) \subseteq [-\pi, \pi) \times (-\infty, t]$ relates to past events and is called an ambit set, $f_t(\cdot; \phi)$ is a deterministic weight function (assumed to be suitable for the integral to exist) and Z is a Lévy basis on $[-\pi, \pi) \times \mathbb{R}$. The weight functions and the ambit sets must be defined cyclically in the angle such that the radial function $R_t(\phi)$ becomes cyclic. In the following, all angular calculations are regarded as cyclic.

The mean growth rate becomes, cf. Appendix A,

$$\mathbb{E}(\frac{\partial}{\partial t} R_t(\phi)) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi; \phi) \mathbb{E}(Z'(\xi)) \mu(d\xi).$$

In the special case where Z is a zero mean Gaussian Lévy basis, $\mu_t(\phi)$ is indeed the mean growth rate at time t in direction ϕ. In other cases, $\mu_t(\phi)$ must be chosen such that the mean growth rate becomes as desired. There is a large literature on deterministic modelling of growth. A classical example is Gompertz growth rate specified by

$$\mathbb{E}(\frac{\partial}{\partial t} R_t(\phi)) = \mu_t = \kappa_0 \exp[\frac{\eta}{\gamma}(1 - \exp(-\gamma t))] \eta \exp(-\gamma t)$$

cf. e.g. **?**

The ambit set $A_t(\phi)$ plays a key role in this modelling approach and affects the degree of dependence on the past. The extend of this dependence may be specified by the minimal time lag $T(t)$ such that

$$A_t(\phi) \subseteq [-\pi, \pi) \times [t - T(t), t], \quad \phi \in [-\pi, \pi).$$

For an illustration, see Figure **??**. Note that it follows from the fact that Z is independently scattered that the random growth rates at time t and t' are independent if $\min(t, t') < \max(t - T(t), t' - T(t'))$. The dependence of the ambit set $A_t(\phi)$ on time t and location ϕ will depend on the specific growth process to be modelled. A number of examples are given below. The induced correlation structure will be discussed in more detail in Section **??**. A discrete version of (**??**) with a Gaussian Lévy basis is discussed in **?**.

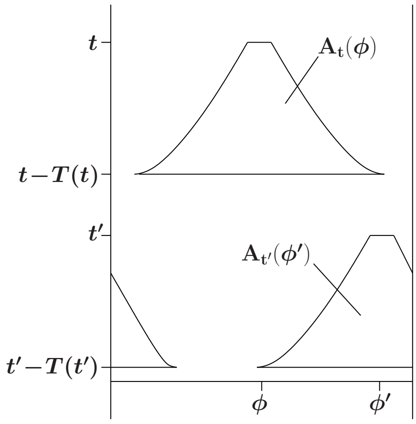

Figure 4: Two ambit sets $A_t(\phi)$ and $A_{t'}(\phi')$. Note the cyclic definition in the angle.

For the interpretation of (**??**) as a growth model it is important to represent the ambit set as a stochastic subset $\tilde{A}_t(\phi)$ of the object, as explained in the introduction for the particular case of a Poisson Lévy basis. This is possible if the stochastic time transformation $t \to R_t(\phi)$ is non-decreasing for each $\phi \in [-\pi, \pi)$.

Under (**??**), the induced model for $R_t(\phi)$ will be of the same linear form since

$$\begin{aligned} R_t(\phi) &= R_0(\phi) + \bar{\mu}_t(\phi) + \int_0^t \int_{A_s(\phi)} f_s(\xi; \phi) Z(d\xi) ds \\ &= R_0(\phi) + \bar{\mu}_t(\phi) + \int_{\bar{A}_t(\phi)} \bar{f}_t(\xi; \phi) Z(d\xi), \end{aligned} \tag{10}$$

where R_0 is the radial function at time $t = 0$,

$$\bar{\mu}_t(\phi) = \int_0^t \mu_s(\phi) ds,$$

$$\bar{A}_t(\phi) = \bigcup_{0 \le s \le t} A_s(\phi),$$

and

$$\bar{f}_t(\xi; \phi) = \int_0^t \mathbf{1}_{A_s(\phi)}(\xi) f_s(\xi; \phi) ds. \tag{11}$$

Note that the ambit sets associated with the radial function itself are increasing

$$t \le t' \Rightarrow \bar{A}_t(\phi) \subseteq \bar{A}_{t'}(\phi).$$

Furthermore, if we use the following notation

$$R_{\le t} = \{R_{t'}(\phi) : t' \le t, \phi \in [-\pi, \pi)\}$$
$$R_{>t} - R_t = \{R_{t'}(\phi) - R_t(\phi) : t' > t, \phi \in [-\pi, \pi)\}$$

then, because Z is independently scattered,

$$R_{\le t - T(t)} \text{ and } R_{>t} - R_t \text{ are independent.}$$

.

The representation (**??**) is, of course, not unique. If, in particular,

$$A_t(\phi) = B_t \cap C_\phi \tag{12}$$

then

$$\bar{A}_t(\phi) = \bar{B}_t \cap C_\phi,$$

where

$$\bar{B}_t = \bigcup_{0 \le s \le t} B_s,$$

and we may choose, instead of (**??**),

$$\bar{f}_t(\xi;\phi) = \int_0^t \mathbf{1}_{B_s}(\xi) f_s(\xi;\phi) ds.$$

In some cases it might be more natural to formulate the model in terms of the time derivative of $\ln(R_t(\phi))$,

$$\frac{\partial}{\partial t}(\ln(R_t(\phi)) = \mu_t(\phi) + \int_{A_t(\phi)} f_t(\xi;\phi) Z(d\xi).$$

In this case the induced model is a Lévy growth model of exponential form

$$R_t(\phi) = R_0(\phi) \exp\left(\bar{\mu}_t(\phi) + \int_{\bar{A}_t(\phi)} \bar{f}_t(\xi;\phi) Z(d\xi)\right).$$

The choice of the Lévy basis Z, the ambit sets $A_t(\phi)$, the weight functions $f_t(\xi;\phi)$ and $\mu_t(\phi)$ completely determines the growth dynamics. These four ingredients can be chosen arbitrarily and independently which results in a great variety of different growth dynamics. We will now give a number of examples.

Example 3. Consider a Lévy growth model for the time derivative of the radial function

$$\frac{\partial}{\partial t} R_t(\phi) = Z(A_t(\phi)),$$

where Z is a Poisson Lévy basis with intensity measure concentrated on $[-\pi, \pi) \times \mathbb{R}_+$ of the form

$$\mu(d\xi) = g(s) ds d\theta, \quad \xi = (\theta, s).$$

Note that the corresponding point process in the Euclidean plane

$$\{(s\cos\theta, s\sin\theta):\ (\theta, s) \text{ is a support point of } Z\}$$

constitutes a Poisson point process with intensity measure

$$\tilde{\mu}(dx) = \frac{g(\|x\|)}{\|x\|}dx,\ x \in \mathbb{R}^2.$$

In particular, if $g(s) = as$, $a > 0$, then Poisson point process in the plane is homogeneous.

The ambit sets are of the form

$$A_t(\phi) = \{(\theta, s) : |\theta - \phi| \leq \frac{\Theta}{s}, \max(0, t-T) \leq s \leq t\}.$$

Represented as subsets of the Euclidean plane

$$\{(s\cos\theta, s\sin\theta):\ (\theta, s) \in A_t(\phi)\}$$

they will as $t \to \infty$ approach rectangles of side lengths 2Θ and T, cf. Figure **??**

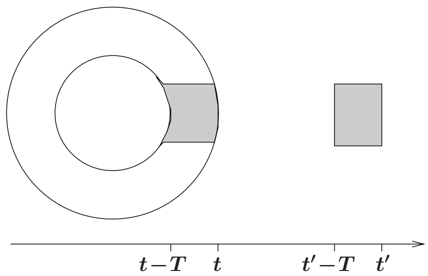

Figure 5: Planar illustrations of the ambit sets.

Note that we can write the ambit set as

$$A_t(\phi) = B_t \cap C_\phi,$$

where

$$B_t = \{(\theta, s) : \max(0, t-T) \leq s \leq t\},$$

and

$$C_\phi = \left\{(\theta, s) : s \geq \frac{\Theta}{\pi}, |\theta - \phi| \leq \frac{\Theta}{s}\right\} \cup \left\{(\theta, s) : 0 \leq s \leq \frac{\Theta}{\pi}\right\}.$$

The mean growth rate at time t and in direction ϕ is for $t > T + \frac{\Theta}{\pi}$

$$\mu(A_t(\phi)) = 2\Theta \int_{t-T}^{t} \frac{g(s)}{s} ds.$$

If $g(s) = as$, $a > 0$, the mean growth rate is constant.

The simulations shown in Figure **??** have been performed with $g(s) = 10s$, $T = 1$ and $\Theta = 1/2$. □

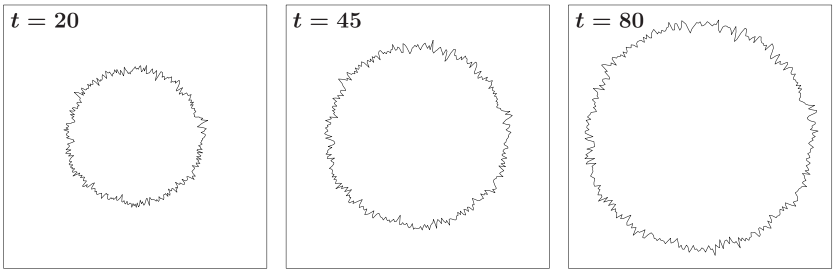

Figure 6: Simulation of a Lévy growth model for the derivative of the radial function at times $t = 20$, 45, 80. The underlying Lévy basis is Poisson.

Example 4. The size of the ambit sets plays an important role in the control of the local and global fluctuations of the boundary of the object Y_t. As an example, let us consider a Lévy growth model of the form

$$R_t(\phi) = \mu_t + Z(A_t(\phi)), \tag{13}$$

where

$$A_t(\phi) = \{(\theta, s) : \ |\theta - \phi| \leq \Theta(s), t - T(t) \leq s \leq t\}.$$

In Figure ??, simulations are shown under this model, using a normal Lévy basis with

$$Z(A) \sim N(0, \sigma^2 \mu(A)), \quad A \in \mathcal{B},$$

and μ equal to the Lebesgue measure on $\mathcal{R}$. The simulations are based on a discretisation of Z on a grid with $\Delta t = 1$ and $\Delta\phi = \frac{2\pi}{1000}$. The upper and lower row of Figure ?? show simulations for two choices of angular extension of the ambit set at three different time points. The angular extension of the ambit set is $\Theta(s) = \frac{\pi}{100}$ for the upper row, while $\Theta(s) = \frac{\pi}{5}$ for the lower row. For the smaller angular extension we observe localised fluctuations of the profiles, but the global appearance is circular. For the larger angular extension the fluctuations are on a much larger scale and the global appearance is more variable. □

Example 5. In this example, we study a model as the one described in Example ??, but now with a Gamma Lévy basis. The model equation is

$$R_t(\phi) = \tilde{\mu}_t + Z(A_t(\phi)), \tag{14}$$

where $A_t(\phi)$ is defined as in Example ??,

$$Z(A) \sim \Gamma(\beta\mu(A), \alpha), \quad A \in \mathcal{B},$$

and μ is the Lebesgue measure on $\mathcal{R}$. Note that $\mu(A_t(\phi))$ does not depend on ϕ. The parameters α, β and $\tilde{\mu}_t$ are chosen such that $\mathbb{E}(R_t(\phi))$ and $\mathbb{V}(R_t(\phi))$ are the same as in the previous example, cf. Appendix A. Accordingly, the parameters are chosen such that

$$\tilde{\mu}_t = \mu_t - \sigma\sqrt{\beta}\mu(A_t(0)),$$
$$\alpha = \sqrt{\frac{\beta}{\sigma^2}}.$$

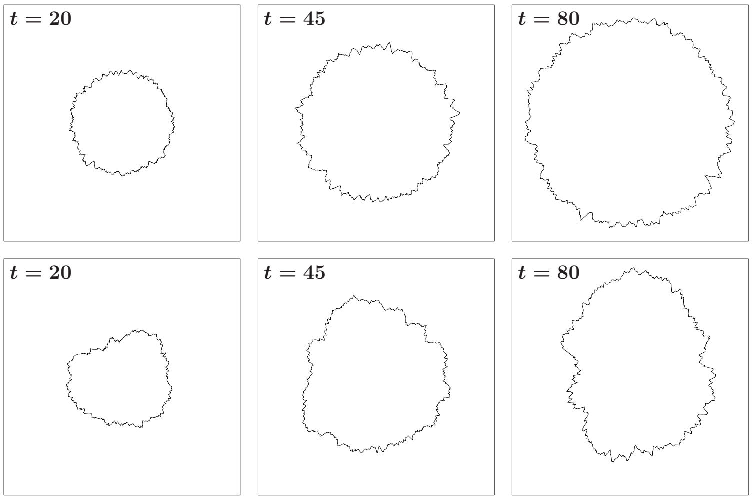

Figure 7: Simulation of the linear Lévy growth model (**??**) at time points $t = 20, 45, 80$, using a Gaussian Lévy basis. The upper row and lower row show simulations of two choices of the angular extension of the ambit set $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Otherwise the parameters of the simulation are $\mu_{20} = 16$, $\mu_{45} = 24$, $\mu_{80} = 32$, $\sigma^2 = 1$ and $T(t) = t/5$.

The only free parameter is $\beta > 0$, determining the skewness of the Gamma distribution of $Z(A_t(\phi))$. For large values of β, the distribution will resemble the Gaussian distribution.

The resulting simulations for $\beta = 1$ are shown in the upper and lower row of Figure **??** for the two choices of angular extension of the ambit set, $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Note that more sudden outbursts are seen compared to the previous example. □

Example 6. In Figure **??**, we show simulations from the Lévy growth model

$$R_t(\phi) = f(\phi)\big(\mu_t + Z(A_t(\phi))\big), \tag{15}$$

where μ_t, $A_t(\phi)$ and Z are specified as in Example **??** and

$$f_t(\phi) = 0.35 \exp\left(\frac{|\phi - \pi|}{\pi}\right).$$

Clearly the growth of the object is asymmetric. The weight function $f_t(\phi)$ puts more weight on the angle $\phi_0 = 0$. □

4 The induced covariance structure

The Lévy based growth models induce new flexible models for space-time covariance functions on the circle, as we shall see in this section where we will derive expressions for $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ under various assumptions on the Lévy basis Z, the ambit sets $A_t(\phi)$ and the weight functions $f_t(\xi;\phi)$. We will concentrate on the Lévy growth model (**??**) of linear form for R_t. Since we now are interested in covariances it suffices to look at the following model equation

$$R_t(\phi) = \int_{A_t(\phi)} f_t(\xi;\phi) Z(d\xi),$$

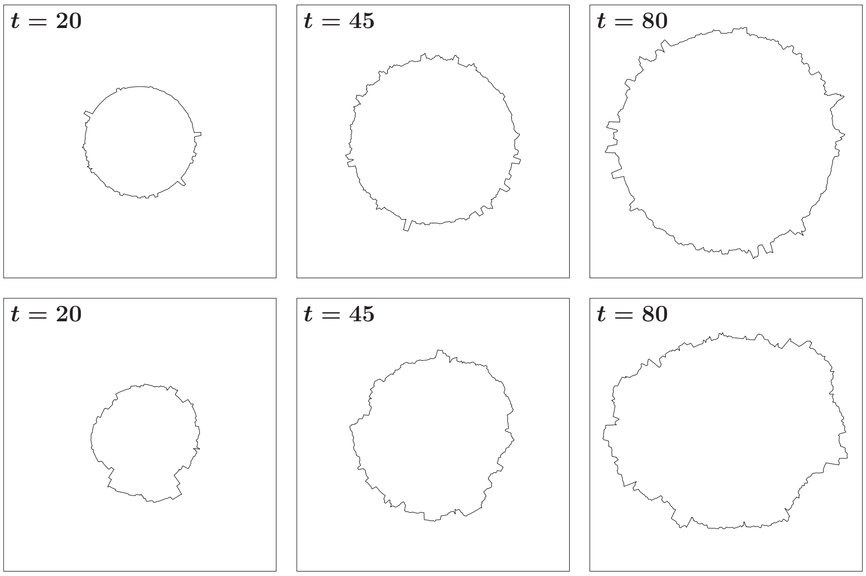

Figure 8: Simulation of the linear Lévy growth model (**??**) at time points $t = 20, 45, 80$, using a Gamma Lévy basis. The upper row and lower row show simulations of two choices of the angular extension of the ambit set $\Theta(s) = \frac{\pi}{100}$ and $\Theta(s) = \frac{\pi}{5}$, respectively. Otherwise, $\beta = 1$ and the remaining parameters are determined by the parameters used in Example **??**.

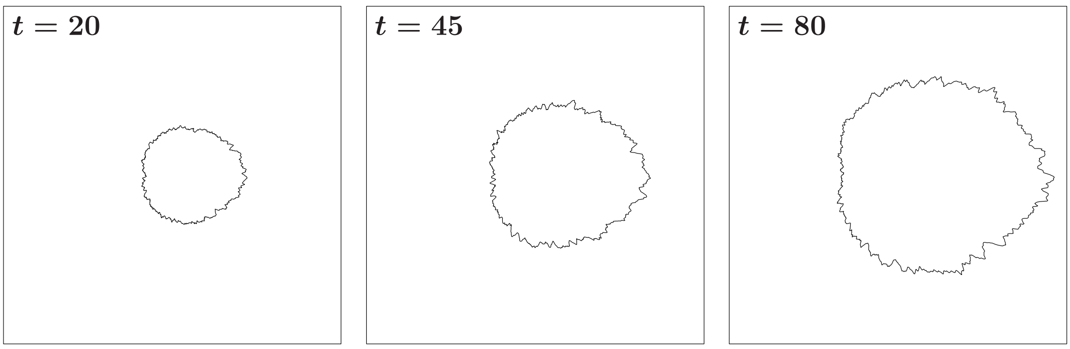

Figure 9: Simulation of the model (**??**) at time points $t = 20, 45, 80$, using a Gaussian Lévy basis, with parameters as specified in Example **??**. The weight function is given by $f_t(\phi) = 0.35 \exp(\frac{|\phi - \pi|}{\pi})$.

where we, for simplicity, have omitted the bar on the ambit set and weight function. The covariance structure of $R_t(\phi)$ is then given by, cf. Appendix A,

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = \int_{A_t(\phi)\cap A_{t'}(\phi')} f_t(\xi;\phi) f_{t'}(\xi;\phi')\mathbb{V}(Z'(\xi))\mu(d\xi). \tag{16}$$

For growth models of exponential form, (**??**) holds for the log-transformed radial function.

Throughout this section, we will assume that

$$\begin{aligned} A_t(\phi) &= (\phi, 0) + A_t(0), \\ f_t(\xi;\phi) &= f_t(|(\theta-\phi|, s); 0), \\ \mathbb{V}(Z'(\xi))\mu(d\xi) &= g(s)dsd\theta \end{aligned} \tag{17}$$

for all $\xi = (\theta, s) \in \mathcal{R}$ and $(\phi, t) \in \mathcal{R}$. These conditions ensure that $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ only depends on the cyclic difference between ϕ and ϕ'. Accordingly, the spatio-temporal process

$$\{R_t(\phi) : t \in \mathbb{R}, \phi \in [-\pi, \pi)\}$$

is second-order stationary in the space coordinate but not necessarily in the time coordinate.

We will first consider the case where the angular extension of the ambit set is the full angular space but the weight functions are arbitrary. Secondly, we consider the case of constant weight functions but quite arbitrary ambit sets.

4.1 Ambit sets with full angular range

In this subsection we consider ambit sets of the form

$$A_t(\phi) = [-\pi, \pi) \times [t - T(t), t].$$

In order to express the formulae as compact as possible, we use in the proposition below the notation $t \cap t'$ for the time points shared by $A_t(\cdot)$ and $A_{t'}(\cdot)$, i.e.

$$t \cap t' = \begin{cases} [\tilde{t}_1, \tilde{t}_2] & \text{if } \tilde{t}_1 \leq \tilde{t}_2 \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\tilde{t}_1 = \max(t - T(t), t' - T(t')) \text{ and } \tilde{t}_2 = \min(t, t').$$

Using this notation we can derive the following convenient and general expression for the covariances.

Proposition 7. Let us assume that the ambit set is of the form $A_t(\phi) = [-\pi, \pi) \times [t - T(t), t]$ for all $(\phi, t) \in \mathcal{R}$ and let

$$f_t(\xi;\phi) = a_0^t(s) + \sum_{k=1}^{\infty} a_k^t(s)\cos(k(\theta-\phi)), \tag{18}$$

$\xi = (\theta, s)$, be the Fourier expansion of the weight function. Then, the spatio-temporal covariances are

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = 2\tau_0(t, t') + \sum_{k=1}^{\infty} \tau_k(t, t')\cos(k(\phi - \phi')), \tag{19}$$

where

$$\tau_k(t,t') = \pi \int_{t\cap t'} a_k^t(s) a_k^{t'}(s) g(s) ds.$$

□

Proof. The proof is straightforward. First note that the actual form (**??**) of the Fourier expansion of the weight function is a consequence of (**??**). We get that

$$\begin{aligned}
&\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) \\
&= \int_{A_t(\phi)\cap A_{t'}(\phi')} f_t(\xi;\phi) f_{t'}(\xi;\phi') \mathbb{V}(Z'(\xi)) \mu(d\xi) \\
&= \pi \left[2\int_{t\cap t'} a_0^t(s) a_0^{t'}(s) g(s) ds + \sum_{k=1}^{\infty} \left(\int_{t\cap t'} a_k^t(s) a_k^{t'}(s) g(s) ds \right) \cos(k(\phi-\phi')) \right].
\end{aligned}$$

□

Example 8. Suppose that the weight function is of the form (**??**) with $a_k^t(s) \equiv 0$ if $k \neq 1$. Then,

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = \pi \cos(\phi - \phi') \int_{t\cap t'} a_1^t(s) a_1^{t'}(s) g(s) ds.$$

Since the covariance is a product of a spatial term and a temporal term, this model is separable, cf. **?** and references therein. The sign of the covariance may be positive or negative. □

Note that according to (**??**) the covariance $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ depends on ϕ and ϕ' only via $|\phi - \phi'|$. For some choices of model parameters, the covariance also becomes stationary in the time coordinate. For instance, if $g(s) = 1$, $T(t) = T$ and $a_k^t(s) = b_k(t-s)$, we have

$$\tau_k(t,t') = \pi \int_{\max(t-t',0)}^{T+\min(t-t',0)} b_k(u) b_k(t'-t+u) du.$$

The induced model (**??**) for the covariance function is not in general separable in the sense that the covariance function can be written as a product of a term depending only on t and t' and a term depending only on ϕ and ϕ'. This may be regarded as a strength of the model because separable covariance functions are often believed to give a too simplistic description of spatio-temporal data, cf. e.g. **?**. The corollary below contains results under such simplifying assumptions.

Corollary 9. Let the assumptions be as in Proposition **??**. Assume that $a_k^t(s) = a_k^t$. Then, the spatial correlations are determined by the weight function f

$$\rho(R_t(\phi), R_t(\phi')) := \frac{\mathrm{Cov}(R_t(\phi), R_t(\phi'))}{\sqrt{\mathbb{V}(R_t(\phi))\mathbb{V}(R_t(\phi'))}} = \frac{2(a_0^t)^2 + \sum_{k=1}^{\infty} (a_k^t)^2 \cos(k(\phi-\phi'))}{2(a_0^t)^2 + \sum_{k=1}^{\infty} (a_k^t)^2}.$$

If, in addition, $a_k^t = b_t c_k$, then the covariance model (**??**) is separable. Furthemore, the spatial correlations $\rho(R_t(\phi), R_t(\phi'))$ do not depend on t, while the temporal correlations are determined by $T(t)$ and the function g,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{\int_{t\cap t'} g(s) ds}{\left[\int_{t-T(t)}^{t} g(s) ds \cdot \int_{t'-T(t')}^{t'} g(s) ds \right]^{1/2}}.$$

□

The covariance model (**??**) provides a possibility for extending stationary covariance functions on the circle (spatial covariance functions) to a spatio-temporal context. When R_t is a stationary process on the circle, its covariance function can be expressed as

$$\mathrm{Cov}(R_t(\phi), R_t(\phi')) = 2\lambda_0^t + \sum_{k=1}^{\infty} \lambda_k^t \cos(k(\phi - \phi')). \tag{20}$$

Such a covariance function can be obtained by choosing the Fourier coefficients of the weight function as

$$a_k^t(s) = a_k^t = \left[\lambda_k^t / \int_{t-T(t)}^{t} g(s)ds \right]^{1/2}.$$

Note that there is still freedom in the modelling by choosing an arbitrary time lag $T(t)$ and function g.

Example 10. The p-order model for a stationary covariance function on the circle, described in **?**, has

$$\lambda_0^t = \lambda_1^t = 0, \ \lambda_k^t = \left[\alpha_t + \beta_t \left(k^{2p} - 2^{2p}\right)\right]^{-1}, \ k = 2, 3, \ldots \ .$$

The model is called a $p-$order model because it can be derived as a limit of discrete $p-$order Markov models defined on a finite, systematic set of angles, cf. **?**. This covariance structure is obtained by choosing

$$a_0^t(s) = a_1^t(s) = 0, \ a_k^t(s) = \left[\pi \int_{t-T(t)}^{t} g(s)ds \right]^{-1/2} \left[\alpha_t + \beta_t \left(k^{2p} - 2^{2p}\right)\right]^{-1/2}, \ k = 2, 3, \ldots \ .$$

If α_t and β_t are proportional, the simplifying assumptions of Corollary **??** are fulfilled. In **?**, this model has been used for the time derivative of the radial function. Only Gaussian Lévy bases are considered and neighbour time points are assumed to be so far apart that the increments can be regarded as independent. The more general approach of the present paper allows for temporal correlations. Under the assumption $a_k^t = b_t c_k$, the temporal correlations are particularly simple. For instance, suppose that $T(t) \equiv 1$ and $t' - 1 \le t \le t'$. Then, we get for $g(s) = ae^{-bs}, a, b > 0$,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{1}{e^b - 1} \left[e^{\frac{1}{2}b(t-t')+b} - e^{-\frac{1}{2}b(t-t')} \right],$$

while for $g(s) = as^{\alpha}$, $a > 0$, $\alpha \ge 1$,

$$\rho(R_t(\phi), R_{t'}(\phi)) = \frac{t^{\alpha+1} - (t'-1)^{\alpha+1}}{\left[(t^{\alpha+1} - (t-1)^{\alpha+1})((t')^{\alpha+1} - (t'-1)^{\alpha+1})\right]^{1/2}}.$$

Only in the first case, the temporal correlations are always stationary. □

4.2 Constant weight functions

In this subsection, we consider the case of constant weight functions. Without loss of generality, we assume that $f_t(\xi;\phi) \equiv 1$ and (**??**) reduces to

$$\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) = \int_{A_t(\phi)\cap A_{t'}(\phi')} \mathbb{V}(Z'(\xi))\mu(d\xi) = \mathbb{V}(Z')\mu(A_t(\phi) \cap A_{t'}(\phi')), \tag{21}$$

where the last equality holds if the Lévy basis is factorisable.

It is not difficult (but sometimes tedious) to find explicit expressions for $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi'))$ for specific choices of ambit sets. In Appendix B, ambit sets of the form

$$A_t(\phi) = B_t \cap C_\phi,$$

where

$$\begin{aligned} B_t &= \{(\theta, s) : \max(0, t - T(t)) \leq s \leq t\}, \\ C_\phi &= \{(\theta, s) : |\phi - \theta| \leq \Theta(s)\}, \end{aligned}$$

are considered. From Appendix B, it is seen that simpler expressions are obtained for the temporal covariances than for the spatial covariances.

Evidently, (**??**) implies that $\mathrm{Cov}(R_t(\phi), R_{t'}(\phi')) \geq 0$ which may be a severe restriction for the spatial covariances. In the proposition below, the spatial covariances are expressed in terms of the function deliminating the ambit set. The proposition gives insight into the class of spatial covariances that can be modelled using this approach.

Proposition 11. Let $\mu(d\xi) = g(s)dsd\theta$ for $\xi = (\theta, s)$. Let us suppose that there exists a continuous function $h_t : [-\pi, \pi) \to \mathbb{R}_+$ with the properties

$$\begin{aligned} &h_t(\phi) = h_t(-\phi) \\ &h_t \text{ is decreasing on } [0, \pi] \\ &h_t(0) = t \end{aligned} \tag{22}$$

such that

$$A_t(0) = \{(\theta, s) : h_t(\pi) \leq s \leq h_t(\theta)\},$$

cf. Figure **??**. Let

$$\bar{h}_t(\phi) = \int_0^{h_t(\phi)} g(s)ds.$$

Then, if the Fourier expansion of $\bar{h}_t$ is ($\bar{h}_t(\phi) = \bar{h}_t(-\phi)$)

$$\bar{h}_t(\phi) = \sum_{k=0}^{\infty} \gamma_k^t \cos(k\phi), \tag{23}$$

then

$$\mu(A_t(0) \cap A_t(\phi)) = \sum_{k=0}^{\infty} \lambda_k^t \cos(k\phi), \tag{24}$$

where

$$\begin{aligned} \lambda_0^t &= \sum_{k \text{ odd}} [\pi - \frac{8}{\pi k^2}]\gamma_k^t - \pi \sum_{k \text{ even}} \gamma_k^t \\ \lambda_j^t &= \frac{16}{\pi} \sum_{k \text{ odd}} \frac{1}{(2j)^2 - k^2} \gamma_k^t, \quad j = 1, 2, \ldots. \end{aligned}$$

□

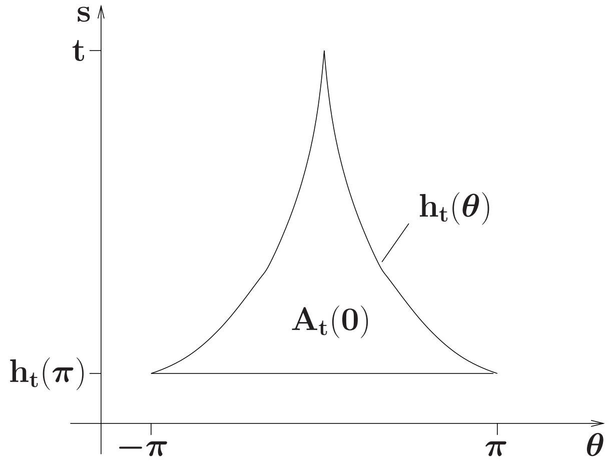

Figure 10: Illustration of the ambit set $A_t(0)$ bounded by the function h_t, cf. (**??**).

Proof. It is not difficult to show that

$$\mu(A_t(0) \cap A_t(\phi)) = 2\int_{-\pi}^{-\pi+\frac{\phi}{2}} \bar{h}_t(\theta)d\theta + 2\int_{\frac{\phi}{2}}^{\pi} \bar{h}_t(\theta)d\theta - 2\pi\bar{h}_t(\pi), \quad \phi \in [0,\pi). \tag{25}$$

Using (**??**), we find

$$\mu(A_t(0) \cap A_t(\phi)) = \begin{cases} -4\sum_{k \text{ odd}} \frac{\gamma_k^t}{k} \sin(k\frac{\phi}{2}) + 2\pi \sum_{k \text{ odd}} \gamma_k^t - 2\pi \sum_{k \text{ even}} \gamma_k^t & \text{if } \phi \in [0,\pi] \\ 4\sum_{k \text{ odd}} \frac{\gamma_k^t}{k} \sin(k\frac{\phi}{2}) + 2\pi \sum_{k \text{ odd}} \gamma_k^t - 2\pi \sum_{k \text{ even}} \gamma_k^t & \text{if } \phi \in [-\pi,0]. \end{cases}$$

The result is now obtained by deriving a Fourier expansion of the latter expression and comparing with (**??**). □

Example 12. In the particular case where $g(s) = 1$ and

$$\bar{f}_t(\phi) = f_t(\phi) = \gamma_0^t + \gamma_1^t \cos\phi$$

we find

$$\lambda_0^t = [\pi - \frac{8}{\pi}]\gamma_1^t - \pi\gamma_0^t$$
$$\lambda_j^t = \frac{16}{\pi}\frac{1}{(2j)^2-1}\gamma_1^t, \quad j = 1,2,\ldots.$$

It follows that

$$(\lambda_j^t)^{-1} = \alpha_t + \beta_t j^2, \quad j = 1,2,\ldots, \tag{26}$$

where $\alpha_t = -\pi/(16\gamma_1^t)$ and $\beta_t = \pi/(4\gamma_1^t)$. Under the assumption of a normal Lévy basis, (**??**) is a special case of the $p-$order model considered in **?** with $p = 1$ and α proportional to β. Note that the requirements (**??**) implies that $\gamma_0^t = t - \gamma_1^t$ and $\gamma_1^t > 0$. It does not seem to be possible to obtain p-order models with $p > 1$, using this approach. □

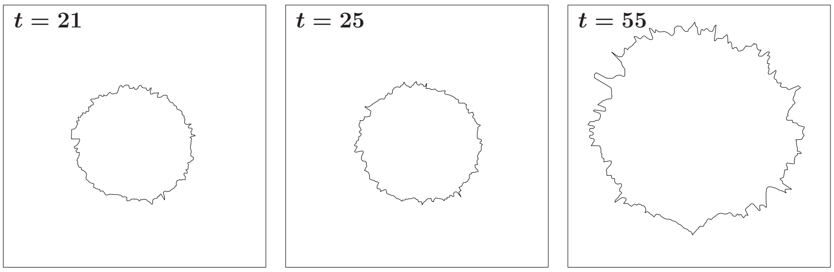

Figure 11: Profiles of a growing brain tumour in vitro at time points $t = 21, 25, 55$.

5 An application to tumour growth

In **?**, snapshots of a growing brain tumour in vitro have been analysed, using the approach described in this paper, see Figure **??**. The data were first studied in **?**.

A detailed initial analysis *I will insert some more details* showed negative spatial covariances and a need for modelling both small and large scale fluctuations in the growth process. The model used was an exponential Lévy growth model of the form

$$R_t(\phi) = \exp\left\{ \mu_t + \alpha(t) \int_{t-T(t)}^{t-t_0(t)} \int_{-\pi}^{\pi} \cos(\phi - \theta) Z(dsd\theta) + \beta(t) \int_{t-t_0(t)}^{t} \int_{\phi - h_t(s-t+t_0(t))}^{\phi + h_t(s-t+t_0(t))} Z(dsd\theta) \right\}. \tag{27}$$

Here h_t is a deterministic and monotonically decreasing function defined on $[0, t_0(t)]$, satisfying $h_t(t_0(t)) = 0$. Accordingly, the weight function is on the form

$$f_t(\xi; \phi) = \alpha(t) \cos(\phi - \theta) \mathbf{1}_{[t-T(t), t-t_0(t)]}(s) + \beta(t) \mathbf{1}_{[t-t_0(t), t]}(s) \mathbf{1}_{[0, h_t(s-t+t_0(t))]}(|\phi - \theta|).$$

The associated ambit set is shown in Figure **??**. In **?**, a Gaussian Lévy basis has been used

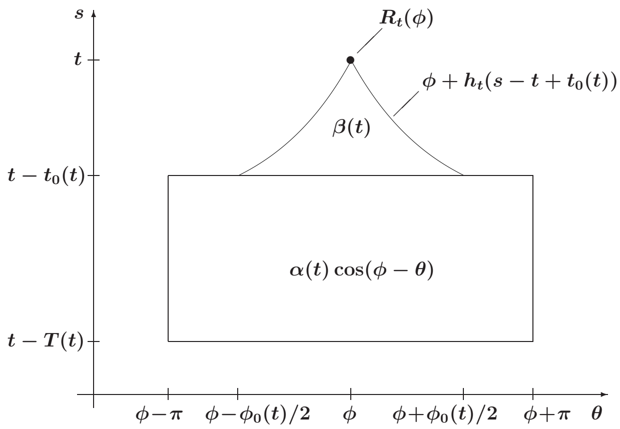

Figure 12: The ambit set $A_t(\phi)$ for the model defined by (**??**).

and the function h_t was assumed to be of the form

$$h_t(s) = \frac{\phi_0(t)}{2} - \frac{\phi_0(t)}{2t_0(t)} s, \quad s \in [0, t_0(t)].$$

The parameters of the model (**??**) were estimated by the method of moments, using the results given in Appendix B. *could be expanded a bit, what about estimation of to(t) and T(t)* The estimated parameters are given in Table **??** and a simulation under the model is shown in Figure **??**.

t	$T(t)$	$t_0(t)$	$\alpha(t)$	$\beta(t)$	$\phi_0(t)$
21	21	19	0.04	−0.033	0.19
25	25	17	0.02	−0.033	0.19
55	18	4	0.01	−0.067	0.23

Table 2: The estimated parameters for the model (**??**), using a Gaussian Lévy basis.

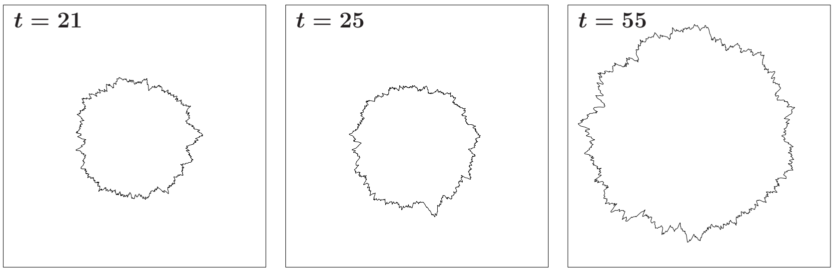

Figure 13: Simulation of the model (**??**) for time points $t = 21, 25, 55$, using a Gaussian Lévy basis.

Here we have studied the use of Gamma and inverse Gaussian Lévy bases. Simulations under the latter basis are shown in Figure **??**. The inverse Gaussian Lévy basis is chosen such that $\mathbb{E}(R_t(\phi))$ and $\mathbb{V}(R_t(\phi)$ are the same as in the case where a Gaussian basis is used. This means that if $Z(A) \sim \mathrm{IG}(\eta\mu(A), \gamma)$, where μ is the Lebesgue measure, we get that $\eta = \gamma^3$. The upper row of Figure **??** shows simulations where $\eta = 316$ and the lower row shows a simulation where $\eta = 5$. For $\eta = 316$ the inverse Gaussian Lévy basis provides fits of a similar quality as the normal basis but for $\eta = 5$, more outburst are observed as is the case for the data. The difference is due to the fact that the inverse Gaussian distribution has heavier right tails for the latter choice of parameters.

6 Discussion

6.1 Estimation of model parameters

Estimation of a set, moment estimation, MLE?, t0(t) og T(t) are difficult to determine

A Fourier expansion of the radial function is useful when studying the shape of the growing object, cf. e.g. **?** and **?**. Let us consider the Fourier coefficients of $R_t(\phi)$,

$$A_k^t = \frac{1}{\pi}\int_{-\pi}^{\pi} R_t(\phi)\cos(k\phi)d\phi, \quad B_k^t = \frac{1}{\pi}\int_{-\pi}^{\pi} R_t(\phi)\sin(k\phi)d\phi$$

$k = 0, 1, \dots$. Under the assumptions of Proposition **??** it can be shown that

$$A_k^t = \int_{-\pi}^{\pi}\int_{t-T(t)}^{t} a_k^t(s)\cos(k\theta)Z(d\theta ds), \quad B_k^t = \int_{-\pi}^{\pi}\int_{t-T(t)}^{t} a_k^t(s)\sin(k\theta)Z(d\theta ds),$$

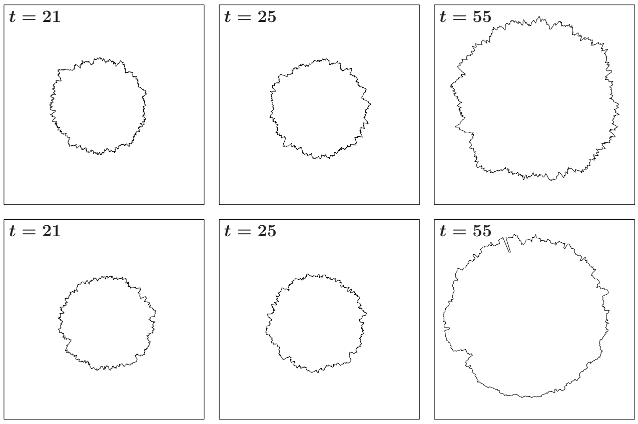

Figure 14: Simulations of the model (**??**) for time points $t = 21, 25, 55$, using an Inverse Gaussian Lévy basis with $\eta = 316$ (upper row) and $\eta = 5$ (lower row).

so the Fourier coefficients also follow a linear Lévy growth model. It can also be shown that for $k \neq j$, $t, t' \geq 0$,

$$\mathrm{Cov}(A_k^t, A_j^{t'}) = \mathrm{Cov}(B_k^t, B_j^{t'}) = \mathrm{Cov}(A_k^t, B_j^{t'}) = 0,$$

and

$$\mathrm{Cov}(A_k^t, A_k^{t'}) = \mathrm{Cov}(B_k^t, B_k^{t'}) = \tau_k(t, t'),$$

where $\tau_k(t, t')$ is given in Proposition **??**.

In the case where Z is a Gaussian Lévy basis, this means that $\{A_k^t\}_{t \in \mathbb{R}_+}$ and $\{B_k^t\}_{t \in \mathbb{R}_+}$ $k = 0, 1, \ldots$, are independent Gaussian stochastic processes with covariance functions $\tau_k(t, t')$. If one observes A_k^t and B_k^t, for some time points $t = t_1, \ldots, t_n$, and some orders $k = 1, \ldots, K_t$, the likelihood function is very tractable.

6.2 Related models

Analogies to MA-processes, Autoregressive analogue?

6.3 The quality of the fit

It should be noted that all the profiles simulated under the model (**??**) using the Lévy basis mentioned in this section show somewhat more fluctuations on a local scale than the observed profiles. At present, we do not know whether this feature is caused by non-perfect model selection and estimation of parameters or artefacts due to the discretisation in the simulation procedure.

6.4 Asymptotics

asymptotic shapes

6.5 New models for covariance functions

insert a number of relevant references, for instance stein and ma(2005)

7 Conclusion

will follow soon

8 Acknowledgements

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9 Appendix A

In this appendix, we summarise formulae for mean values and higher order moments for a random variable expressible in terms of an integral with respect to a Lévy basis.

More specifically, we consider a random variable $X_t(\sigma)$, depending on time t and position σ. In the following, we will assume that $(\sigma, t) \in \mathcal{R} = \mathcal{S} \times \mathbb{R}$, where $\mathcal{S} \subset \mathbb{R}^d$. The random variable $X_t(\sigma)$ is given by

$$X_t(\sigma) = \int_{A_t(\sigma)} f_t(\xi;\sigma) Z(d\xi).$$

We let

$$\tilde{X}_t(\sigma) = \exp(X_t(\sigma)).$$

Using the key relation (**??**), we find

$$\begin{aligned} \mathbb{E}(X_t(\sigma)) &= \int_{A_t(\sigma)} f_t(\xi;\sigma) \mathbb{E}(Z'(\xi)) \mu(d\xi) && (28) \\ \mathbb{V}(X_t(\sigma)) &= \int_{A_t(\sigma)} f_t^2(\xi;\sigma) \mathbb{V}(Z'(\xi)) \mu(d\xi). \end{aligned}$$

The covariances are of the form

$$\mathrm{Cov}(X_t(\sigma), X_{t'}(\sigma')) = \int_{A_t(\sigma) \cap A_{t'}(\sigma')} f_t(\xi;\sigma) f_{t'}(\xi;\sigma') \mathbb{V}(Z'(\xi)) \mu(d\xi). \tag{29}$$

If the weight function is constant, $f_t(\xi;\sigma) \equiv f$, and if the Lévy basis Z is factorisable, then (**??**) reduces to

$$\mathrm{Cov}(X_t(\sigma), X_{t'}(\sigma')) = f^2 \mathbb{V}(Z') \mu(A_t(\sigma) \cap A_{t'}(\sigma')). \tag{30}$$

In this case, the covariance structure only depends on the μ-measure of the intersection of the two ambit sets.

Equation (**??**) enables us to calculate arbitrary mixed n-order moments of $\tilde{X}_t(\sigma)$. If the moments are finite, then

$$\mathbb{E}\left(\tilde{X}_{t_1}(\sigma_1) \cdot \ldots \cdot \tilde{X}_{t_n}(\sigma_n)\right) = \exp\left(\int_{\mathcal{R}} K\left(\sum_{j=1}^{n} f_{t_j}(\xi;\sigma_j) \mathbf{1}_{A_{t_j}(\sigma_j)}(\xi) \ddagger Z'(\xi)\right) \mu(d\xi)\right). \tag{31}$$

The corresponding expressions for the mixed n-order moments of $X_t(\sigma)$ are obtained from

$$\mathbb{E}(X_{t_1}(\sigma_1) \cdot \ldots \cdot X_{t_n}(\sigma_n)) = \frac{\partial^n}{\partial\lambda_1 \cdot \ldots \cdot \partial\lambda_n} \mathbb{E}\left(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1) \cdot \ldots \cdot \tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\right)\Big|_{\lambda_1 = \ldots = \lambda_n = 0} \tag{32}$$

where

$$\mathbb{E}\left(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\right)=\exp\left(\int_{\mathcal{R}}K\left(\sum_{j=1}^{n}\lambda_j f_{t_j}(\xi;\sigma_j)\mathbf{1}_{A_{t_j}(\sigma_j)}(\xi)\ddagger Z'(\xi)\right)\mu(d\xi)\right). \quad (33)$$

The relative second order moments of $\tilde{X}_t(\sigma)$ has a particularly attractive form

$$\frac{\mathbb{E}(\tilde{X}_t(\sigma)\tilde{X}_{t'}(\sigma'))}{\mathbb{E}(\tilde{X}_t(\sigma))\mathbb{E}(\tilde{X}_{t'}(\sigma'))}=\exp\left(\int_{A_t(\sigma)\cap A_{t'}(\sigma')}g(\xi;t,t',\sigma,\sigma')\mu(d\xi)\right), \quad (34)$$

where

$$\begin{aligned}&g(\xi;t,t',\sigma,\sigma')\\&=K((f_t(\xi;\sigma)+f_{t'}(\xi;\sigma'))\ddagger Z'(\xi))-K(f_t(\xi;\sigma)\ddagger Z'(\xi))-K(f_{t'}(\xi;\sigma')\ddagger Z'(\xi)).\end{aligned}$$

In the simple case where the weight functions are constant, i.e. $f_t(\xi;\sigma)\equiv f$ for all $(\sigma,t)\in\mathcal{R}$ and $\xi\in\mathcal{R}$, and where the underlying Lévy basis is factorisable, $Z'(\xi)=Z'$, (**??**) reduces to

$$\exp\left(\overline{C}\mu(A_t(\sigma)\cap A_{t'}(\sigma'))\right), \quad (35)$$

where $\overline{C}=K(2f\ddagger Z')-2K(f\ddagger Z')$. For a factorisable Lévy basis Z and a constant weight function, one can express

$$\frac{\mathbb{E}\left(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\cdot\ldots\cdot\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\right)}{\mathbb{E}\left(\tilde{X}_{t_1}^{\lambda_1}(\sigma_1)\right)\cdot\ldots\cdot\mathbb{E}\left(\tilde{X}_{t_n}^{\lambda_n}(\sigma_n)\right)} \quad (36)$$

in terms of different overlaps of the corresponding ambit sets (**?**).

10 Appendix B

In this Appendix we will assume that the measure μ is on the form

$$\mu(d\theta,ds)=g(s)dsd\theta.$$

We will study intersections of ambit sets

$$A_t(\phi)=B_t\cap C_\phi,$$

where

$$\begin{aligned}B_t&=\{(\theta,s):\max(0,t-T(t))\le s\le t\},\\C_\phi&=\{(\theta,s):|\phi-\theta|\le\Theta(s)\},\end{aligned}$$

so the size and shape of $A_t(\phi)$ does not depend on ϕ. This implies that the measure of the intersection $\mu(A_t(\phi)\cap A_{t'}(\phi'))$ only depends on the angles ϕ and ϕ' via their cyclic difference. We therefore focus on studying

$$\mu(A_t(0)\cap A_t(\phi))\quad\text{and}\quad\mu(A_t(0)\cap A_{t+u}(0)),$$

where $u \geq 0$, concentrating on the spatial and temporal covariances/correlations, respectively.

Let $t, u \geq 0$ and assume that $T(t+u) - T(t) \leq u$, then

$$\mu(A_t(0) \cap A_{t+u}(0)) = \begin{cases} 2\int_{t^*}^{t} \Theta(s)g(s)ds & \text{if } u \leq T(t+u), \\ 0 & \text{otherwise} \end{cases}$$

where $t^* = \max(0, t+u-T(t+u))$. In this case one can get various temporal covariance/correlation structures, depending on the choise of T, Θ and g. A few examples are given in Table **??**.

Parameters	$2\Theta(s)g(s)$	$\mu(A_t(0) \cap A_{t+u}(0)$
$c > 0$	c	$c(T(t+u) - u)$
$a, b > 0$	ae^{-bs}	$\frac{ae^{-bt}}{b}(e^{-b(u-T(t+u))} - 1)$
$a > 0,\ \alpha > 1$	as^{α}	$(\alpha-1)^{-1}(t^{\alpha+1} - (t+u-T(t+u))^{\alpha+1})$

Table 3: Explicit expressions for the integral $\mu(A_t(0) \cap A_{t+u}(0))$ for various choises of $\Theta(s)g(s)$.

Let us now study the covariance related to angular displacements. We get that

$$\mu(A_t(0) \cap A_t(\phi)) = \int_{t^*}^{t} \mathbf{1}_{[0,2\Theta(s)]}(|\phi|)\,(2\Theta(s) - \min(|\phi|, 2\pi - 2\Theta(s)))\,g(s)ds, \tag{37}$$

where $t^* = \max(0, t - T(t))$. Here, the integral depends on the seperate choices of Θ and g. Note that in the simple case where $\Theta(s) \equiv \Theta$, we have

$$\mu(A_t(0) \cap A_t(\phi)) = 0, \quad \text{for } |\phi| \geq 2\Theta,$$

which e.g. implies that for a Lévy growth model of linear or exponential form, the values of the radial function at a fixed time t and different angles ϕ and ϕ' are uncorrelated if $|\phi - \phi'| \geq 2\Theta$. If instead $\Theta(s) = \Theta/s$, we get for $T(t) = t$ and $g(s) = s$

$$\mu(A_t(0) \cap A_t(\phi)) = \begin{cases} \frac{1}{2}\frac{(2\Theta)^2}{2\pi - |\phi|} + \left(2\Theta - \frac{|\phi| t}{2}\right) t, & \text{if } |\phi| \leq \frac{2\Theta}{t} \\ (2\Theta)^2 \frac{\pi}{|\phi|(2\pi - |\phi|)}, & \text{if } \frac{2\Theta}{t} < |\phi| \leq \pi, \end{cases}$$

if $t \geq 2\Theta/\pi$.

Note that for a Lévy growth model of linear form with $\mathbb{E}(R_t(\phi)) \propto 2\Theta t$, the covariance $\mathrm{Cov}(R_t(0), R_t(\phi))$ depends for fixed t on $|\phi|$ via $|\phi| t$ for small $|\phi| t$, which is proportional to the distance between two points in directions 0 and ϕ on the boundary of a disc with radius $\mathbb{E}(R_t(\phi))$.

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